Lecture Notes on Trends, Business Cycles, and Measurement

or

How I Learned to Stop Worrying and Love Macroeconometrics

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Abstract

These lectures notes are intended for use in first and second year applied macro and macroeconometrics courses for PhD students. The first section reviews several issues in decomposing aggregate quantity and price time series into trend and cycle. Macroeconomists estimate these hidden states of the aggregate economy using unobserved components models, linear filters, and/or least squares filtering. The estimated trends and cycles are often employed to compute sample moments of the business cycle to test dynamic stochastic general equilibrium (DSGE) and other macro models. The second section of these notes studies a one-sector stochastic growth model. This canonical real business cycle (RBC) model is built from the maximization problems of a representative firm and household. Solving this problem produces optimality and equilibrium conditions, which are detrended before constructing the steady state of the RBC model and its approximate linearized solution. The linear approximate solution is a tool to generate synthetic samples on which to calculate theoretical moments of the RBC model that can be compared to sample moments of the business cycle.

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1. Introduction

The question of how to test models of the business cycle transmssion mechanism rest, in part, on data. The converse is data alone is insufficient to evaluate business cycles theories. Unfortunately, data often need to be transformed to test these theories and models. A reason is the trends apparent in quantity aggregates and price indexes produced for the national income and product accounts (NIPAs). Trends are measurement error relative to the signal of the business cycle, according to the practice engaged by some macroeconomists.

The NIPAs report GDP, personal consumption expenditures, fixed and intangible investment, imports, exports, government expenditures, and various price deflators that are often upward trending. The top panels of figures 1 and 2 depict this behavior for the logs of U.S. per capita real GDP and its chained weighted price deflator. However, these variables also exhibit transitory movements, or business cycle fluctuations, around the trends as suggested for output growth and inflation in the middle panels of these figures. If the goal is to test business cycle theory, stripping data of trends would seem to be an important task. However, there is no consensus among macroeconomists about the best method(s) to decompose aggregate data into trends and cycles. A sampling of these disparate views is reflected by debates about (*i*) turning point analysis and business cycle analysis by Burns and Mitchell (1946), Koopman (1947), Simkins (1994), King and Plosser (1994), Harding and Pagan (2016), Kulish and Pagan (2019), and Beaudry, Galizia, and Portier (2020) and (*ii*) linear filtering and business cycle analysis by, among others, Prescott (1986), Harvey and Jaeger (1993), King and Rebelo (1993), Cogley and Nason (1995a), Hodrick and Prescott (1997), Canova (1998a, 1998b), Burnside (1998), Gómez (1999), Harvey and Trimbur (2003), Murray (2003), Nelson (2008), Hamilton (2018), and Hodrick (2020). The lack of consensus motivates the first part of these notes to review methods to separate trend from cycle in sample data. The way in which this task is accomplished can affect the outcome of tests of business cycle theories. Hence, the review focuses on the trade offs across several methods used to recover the hidden states of the trend and cycle of an economy.

Measuring business cycle fluctuations in sample data is one aspect of the process of testing theory. Another part builds dynamic stochastic general equilibrium (DSGE) models to compare theoretical business cycle moments to sample moments. A one-sector stochastic growth model is studied in the second part of the notes to keep the discussion manageable. The notes build, solve for optimality conditions, construct the steady state, detrend and linearize these conditions, and compute a linear approximate equilibrium of this real business cycle (RBC) model.

2. Measuring Trends and Business Cycles

Let's begin the discussion by retrieving trends and cycles from aggregate data with a decomposition of real GDP

$$\ln y_t = \tau_t + \varepsilon_t, \tag{2.1}$$

where $\ln y_t$ denotes the natural logarithm of output, τ_t is the trend or permanent component of output, and its transitory or cyclical component is ε_t. This univariate decomposition presumes ε_t is covariance-stationary, but τ_t is not. An implication is the first-difference of $\ln y_t$, $\Delta \ln y_t$ $= \ln y_t - \ln y_{t-1}$, is stationary or first-order intergrated, $\ln y_t \sim I(1)$.

Several methods are available to separate the trend and cycle in equation (2.1). These notes discuss UC models, linear filtering, and least-squares filtering. A UC model specifies the data generating processes (DGPs) of τ_t and ε_t while imposing restrictions to identify innovations to these state variables. Univariate UC models are estimated by Watson (1986), Harvey and Jaeger (1993), Harvey and Koopman (2000), Morley, Nelson, and Zivot (2003), Harvey and Trimbur (2003), Stock and Watson (2007, 2010), Shephard (2013), and Cogley and Sargent (2015). Harvey, Trimbur, and Van Dijk (2007), Creal, Koopman, and Zivot (2010) and Berger, Everaert, and Hauke (2016) engage Bayesian methods to estimate multivariate UC models. Linear filtering depends either on restrictions about the forecast horizons of interest, the underlying autoregressive moving average (ARMA) model, or the relative volatility of cyclical fluctuations to movements in the trend. Whether the underlying DGP generates a finite sample or an infinite span of time series data is key for understanding linear filtering; see Proietti and Harvey (2000), Gómez (1999, 2001), Pollock (2000), Schleicher (2004), Canova (2007, ch. 3), and DeJong and Dave (2011, ch. 6). Least squares filtering applies ordinary least squares (OLS) regressions to actual data, but leads and lags the dependent variable and its regressors to obtain forecast innovations that yield an estimate of the cyclical component. Hamilton (2018) is a recent example of least squares filtering. Nonetheless, UC models, linear filtering, least squares filtering, and least squares smoothing are linked together as shown by, among others, Harvey and Jaeger (1993), Gómez (1999), Harvey and Koopman (2000), and Harvey and Trimbur (2003). These results have implications for interpreting and using different trend-cycle decompositions.

There are other methods available to estimate trends and cycle in aggregate data. Turning point analysis, penalized least squares smoothing, and Markov-switching (MS) models are among the most useful, but time and space constraints suggest the interested reader seek other sources to learn about these methods. Harding and Pagan (2016) is a modern treatment of turning point analysis of business cycles. Penalized least squares smoothing, which should not be confused with least squares filtering, is discussed by Gómez (1999). Hamilton (1989) introduces MS models to macroeconomics. Kim and Nelson (1999) is a good initiation into the econometric modeling of MS in aggregate and financial time series.

2.a *UC Models*

The decomposition (2.1) is the observation or measurement equation of a state space representation of an UC model. An UC model is completed by specifying laws of motion for τ_t and ε_t, which are hidden or latent state variables, and an impulse structure for the innovations to these states. Laws of motion for τ_t can be a random walk (with drift), local level trends, or segmented trend or splines of these processes. The cyclical component ε_t is often in the class of stationary autoregressive-moving average (ARMA) models, which also can be subject to similar breaks. Markov-switching (MS) is a class of time series models that can generate τ_t and/or ε_t. However, MS models are often nonlinear as in Hamilton (1989) and Kim and Nelson (1999).

Morley, Nelson, and Zivot (2003) construct a UC model in which trend output evolves as a random walk with drift

$$\tau_t \;=\; \mu \,+\, \tau_{t-1} \,+\, \sigma_\eta \eta_t, \quad \eta_t \;\sim\; \mathcal{N}(0,\,1), \tag{2.2}$$

where $\mu > 0$ is the deterministic trend growth rate and σ_η scales the volatility of the Gaussian innovation, η_t, to τ_t. The long-run forecast of trend output is $\lim_{k\to\infty} \mathbf{E}_t \left\{\tau_{t+k} - k\mu\right\} \;=\; \tau_t$ because it is a random walk. This carries over to the long-run forecast of $\ln y_t$. Combine the observation equation (2.1) and random walk (2.2) to generate the long-run forecast of output, $\lim_{k\to\infty} \mathbf{E}_t \left\{\ln y_{t+k} - k\mu\right\} = \tau_t$, as the current realization of the random walk trend.

Beveridge and Nelson (1981) are first to link a long-run forecast of an observable to the current realization of a random walk trend. Watson (1986) and Morley, Nelson, and Zivot (2003) point out that a UC model with a random walk state equation ties its long-run forecast of the observable to the Beveridge and Nelson (BN) trend. This forecast is independent of restrictions on ε_t other than it is (conditionally) linear and stationary. Morley (2002) uses this framework and the Kalman filter to construct the BN trend for a stationary finite-order ARMA of $\Delta \ln y_t$.

The transitory component of output is a $\mathrm{AR}(p)$

$$\varepsilon_t \;=\; \sum_{j=1}^{p} \theta \varepsilon_{t-j} \,+\, \sigma_\upsilon \upsilon_t, \quad \upsilon_t \;\sim\; \mathcal{N}(0,\,1), \tag{2.3}$$

in Morley, Nelson, and Zivot (2003), where $\mathbf{E}\varepsilon_t = 0$ and the eigenvalues of the companion matrix

$$\boldsymbol{\Theta} \;\equiv\; \begin{bmatrix} \theta_1 & \theta_2 & \dots & \theta_{p-1} & \theta_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix},$$

are outside the unit circle (*i.e.*, the $\mathrm{AR}(p)$ is stationary).

Morley, Nelson, and Zivot (2003) analyze and estimate their UC model given different identifying assumptions. They set $p = 2$ and use the Kalman filter to estimate the states τ_t and ε_t, produce the predictive likelihood, and employ classical optimization to obtain estimates of μ, θ_1, θ_2, σ_η, and σ_υ. Their estimates of τ_t and ε_t, depend on whether the correlation of the innovations to these states, η_t and υ_t, $\varrho_{\eta,\upsilon}$, is estimated or restricted to zero. When estimates of $\varrho_{\eta,\upsilon} \in (-1, 0)$, trend fluctuations dominate movements in $\ln y_t$.

Another recent application of the UC model of equations (2.1) and (2.2) restricts ε_t to be white noise (*i.e.*, $\theta_j = 0$ for $j = 1, \ldots, p$). Stock and Watson (2007, 2010) and Cogley and Sargent (2015) apply this UC model to estimate the BN trend of inflation. Estimates of trend inflation, which are relatively smooth, are dominated by time-varying heteroscedasticity (*i.e.*, stochastic volatility in the scale volatilities on η_t and υ_t), in the inflation gap, ε_t, in these studies.

Harvey and Jaeger (1993) and Harvey and Trimbur (2003) estimate univariate UC models with different classes of trend and cyclical processes. The former stochastic process is the local level trend. A bivariate local level trend is specified by Harvey and Jaeger (1993) as

$$\begin{aligned} \delta_{1,t} &= \delta_{1,t-1} + \delta_{2,t-1} + \sigma_{\zeta,1}\zeta_{1,t}, \qquad \zeta_{1,t} \sim \mathcal{N}(0, 1), \\ \delta_{2,t} &= \delta_{2,t-1} + \sigma_{\zeta,2}\zeta_{2,t}, \qquad \zeta_{2,t} \sim \mathcal{N}(0, 1), \end{aligned} \tag{2.4}$$

where $\mathbf{E}\left\{\zeta_{1,t+j}\,\zeta_{2,t+s}\right\} = 0$ for all j, s. The level trend $\delta_{1,t}$ is an integrated random walk while its slope $\delta_{2,t}$ is time-varying because it is also a random walk. When $\zeta_{2,t}$ generates a movement in $\delta_{2,t}$, its impact on the slope of $\delta_{1,t}$ lasts forever. The observation equation is

$$\ln y_t = \delta_{1,t} + \vartheta_t + \varpi_t \tag{2.5}$$

for the local level trend-UC model, where ϑ_t is a transitory component and ϖ_t is white noise measurement error $\varpi_t \sim \mathcal{N}(0, \sigma^2_\varpi)$. The transitory component is a stochastic cycle

$$\begin{bmatrix} \vartheta_t \\ \vartheta^*_t \end{bmatrix} = \begin{bmatrix} \rho\cos\kappa_c & \rho\sin\kappa_c \\ -\rho\sin\kappa_c & \rho\cos\kappa_c \end{bmatrix} \begin{bmatrix} \vartheta_{t-1} \\ \vartheta^*_{t-1} \end{bmatrix} + \begin{bmatrix} \sigma_\nu & 0 \\ 0 & \sigma^*_\nu \end{bmatrix} \begin{bmatrix} \nu_t \\ \nu^*_t \end{bmatrix}, \tag{2.6}$$

where $\rho \in (0, 1]$ acts to dampens cycles, $\kappa_c \in (0, \pi]$ is the frequency of cycles in radians ($\pi =$ 3.14159...), $\nu_t \sim \mathcal{N}(0, 1)$, $\nu_t^* \sim \mathcal{N}(0, 1)$, and $\mathbf{E}\left\{\nu_{t+j}\, \nu_{t+s}^*\right\} = 0$ for all j, s. Harvey and Jaeger (1993) note the stochastic cycle (2.6) is observationally equivalent to a ARMA$(2, 1)$ restricted to have complex roots. These restrictions aim to replicate business cycle-like behavior in ϑ_t.

Harvey and Trimbur (2003) generalize the local level trend-UC model of (2.5), (2.4), and (2.6) in two ways. First, they specify $M-1$ level trends that are driven by a common slope trend

$$\begin{aligned} \delta_{1,t} &= \delta_{1,t-1} + \sigma_\zeta \zeta_t, & \zeta_t &\sim \mathcal{N}(0, 1), \\ \delta_{i,t} &= \delta_{i,t-1} + \delta_{i-1,t-1}, & i &= 2, \ldots, M. \end{aligned} \tag{2.7}$$

The stochastic cycle also has M components, but the innovation to $\psi_{i,t}^*$, $\nu_{i,t}^*$, is set to zero for $i = 1, \ldots, M$. The observation equation is

$$\ln y_t = \delta_{m,t} + \psi_{m,t} + \varpi_t \tag{2.8}$$

for the generalized local level trend-UC model. The generalized local level trend-UC model endows $\ln y_t$ with m unit roots. This suggests the generalized local level trend-UC model can be used to obtain estimates of trends and stochastic cycles for any $I(M)$ time series.

A strength of UC models is the explicit character of the assumptions used to identify the decomposition of $\ln_t$ into trend and cycle. These assumptions are clear about the dynamic properties of these hidden state variables. For example, Morley, Nelson, and Zivot (2003) show that estimates of the BN cycle in U.S. output depend on whether $\varrho_{\eta,\upsilon} = 0$ or is estimated. Similarly, Harvey and Trimbur (2003) report estimates of the stochastic cycle $\psi_{M,t}$ of U.S. investment that are sensitive to the choice of M. Hence, a strength of UC models is also a weakness. Misspecification and non-identification introduce biases into estimates of trends and cycles produced by UC models. Bayesian estimation of UC models, as in Harvey, Trimbur, and Van Dijk (2007), is not immune from identification problems. These problems appear in a different guise in Bayesian time series econometrics as discussed by Poirier (1998), among others.

2.b *Linear Filters*

A second approach to calculate trend and cycle is linear filtering. A linear filter solves a mean square error (MSE) minimization problem. This is a consequence of the Riesz–Fischer theorem. It states a (measurable) function on $[-\pi, \pi]$ converges in MSE, given (if and only if) the associated Fourier series does; see Sargent (1987, pp. 249-253) and DeJong and Dave (2011, p. 123). However, the object of choice that solves the minimization problem define different classes of linear filters. This distinction is important for understanding several issues about linear filters. A complete analyses of linear filters is left to, among others, Koopmans (1974), Brillinger (1981), Granger and Newbold (1986), Harvey (1991, 1994), and Hamilton (1994).

Advocates of using linear filters to achieve this decomposition argue as Prescott (1986) does that "business-cycle phenomena ... are nothing more nor less than a certain set of statistical properties of a certain set of important aggregate time series." Prescott states that facts about the business cycle are immutable given a 'smooth' process for the trend. The source of the idea of smooth trend is Lucas (1981). He states, innocently enough, that the trend of a developed economy evolves as a smooth curve that slowly undulates through time. The trend component of a linear filter captures this behavior. Lucas goes on to argue that the business cycle can be represented, 'by a stochastically disturbed difference equation of very low order' about the slowing evolving trend. Some students of the business cycle take this to mean the trend is independent of the econometric problem of measuring the transitory path of an economy. Instead, the trend is defined by the algorithm that separates cycle from trend. Leading examples are the linear filters of Hodrick and Prescott (1997) and Baxter and King (1999).

Linear filters operate to remove the irregular (*i.e.*, temporary) component of the actual time series from its trend. An example of a linear filter is the first difference operator, $\Delta \equiv 1-\mathbf{L}$, $\mathbf{L}$ is the lag operator, $x_{t-1} = \mathbf{L}\, x_t$. It annihilates the permanent component of a difference stationary or $I(1)$ time series. All that remains is a transitory component, which is not necessarily the true cyclical process of x_t. Hence, Δx_t leaves only higher frequency fluctuations, which makes

Δ an example of a high pass filter. Similarly, a low-pass filter removes all but a selected range of lower frequency movements in the data. There are linear filters that annihilate the low and high frequencies. These are referred to as band pass filters.

2.b.i *The Frequency Domain and Spectral Analysis*

Studying linear filters is made easier by introducing several concepts from spectral analysis. Spectral analysis operates in the frequency domain, $\omega \in [0, 2\pi]$, rather than the time domain. The frequency domain works to recover the cyclical properties of time series. These properties are buried in the autocovariances, $\{\gamma_\ell\}_{\ell=-\infty}^{\infty}$, of a times series, x_t. Hence, frequency domain analysis starts with the autocovariance generating function (ACGF). The ACGF of x_t is $g_x(z) = C(z)C(z^{-1})g_u(z) = |C(z)|^2 g_u(z) = \sum_{\ell=-\infty}^{\infty} \gamma_\ell z^\ell$, where $|C(z)|$ is the modulus of $C(z)$, γ_ℓ is the autocovariance of x_t at lag ℓ, its symmetry sets $\gamma_\ell = \gamma_{-\ell}$, and the ACGF of $u_t \sim \mathcal{IID}(0, 1)$ is one because its $\gamma_0 = 1$ and $\gamma_\ell = 0$ for all $\ell \neq 0$.

The next step is to utilize de Moivre's formula. It defines the complex unit circle to be $e^{-i\omega} = \cos(\omega) - i\sin(\omega)$, where $i = \sqrt{-1}$. Setting this equal to z and dividing by 2π yields the population spectrum of x_t, $S_x(\omega) = 0.5\pi^{-1}g_x(e^{-i\omega}) = \left|C(e^{-i\omega})\right|^2 g_u(e^{-i\omega}) = 0.5\pi^{-1}\sum_{\ell=-\infty}^{\infty} \gamma_\ell e^{-i\omega\ell}$. This shows the population spectrum of x_t, $S_x(\omega)$, is a function of autocovariances at any frequency $\omega \in [0, 2\pi]$

$$
\begin{aligned}
S_x(\omega) &= \frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty} \gamma_\ell \left[\cos(\omega\ell) - i\sin(\omega\ell)\right] \\
&= \frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty} \gamma_\ell \left[\cos(\omega\ell) + \cos(-\omega\ell) - i\sin(\omega\ell) - i\sin(-\omega\ell)\right] \\
&= \frac{1}{2\pi}\left[\gamma_0 + 2\sum_{\ell=1}^{\infty} \gamma_\ell \cos(\omega\ell)\right],
\end{aligned}
$$

where the first equality is an implication of the spectral representation theorem and the last equality relies on $\cos 0 = 1$, $\cos\varpi = \cos(-\varpi)$, $\sin 0 = 0$, and $\sin(-\varpi) = -\sin\varpi$.

The population spectrum has several implications that are useful for studying aggregate data. First, the long-run behavior of x_t is described by the population spectrum at frequency zero, $S_x(0)$. Summing the γ_ℓs from $\ell = 0, \ldots, \infty$ yields $S_x(0) = 0.5\pi^{-1}\left[\gamma_0 + 2\sum_{\ell=1}^{\infty}\gamma_\ell\right]$, which is the long-run variance of x_t. Estimators of long-run variances are used, for example, to compute heteroscedastic, autocorrelation consistent (HAC) robust (*i.e*, Newey-West) standard errors for linear and nonlinear regressions. Next, suppose $S(\omega)$ is integrated over some frequencies from $-\omega_a$ to ω_a, $\int_{-\omega_a}^{\omega_a} S_x(\omega)d\omega$. Since the population spectrum is symmetric, it equals $2\int_0^{\omega_a} S_x(\omega)d\omega$. Integrating $S_x(\omega)$ gives the random periodic components that are responsible for the variance of x_t from frequency zero to ω_a. Thus, the population spectrum at frequency ω measures the contributions of periodic or cyclical movements less than ω that explain the variance of x_t.

The sample version of the population spectrum is the sample periodogram. The sample periodogram of x_t is

$$\widehat{S}_x(\omega) = \frac{1}{2\pi}\sum_{\ell=-T+1}^{T-1}\widehat{\gamma}_\ell e^{-i\omega\ell} = \frac{1}{2\pi}\left[\widehat{\gamma}_0 + 2\sum_{\ell=1}^{T-1}\widehat{\gamma}_\ell\cos(\omega\ell)\right],$$

where $\widehat{\gamma}_\ell$ denotes a sample estimate of T autocovariances from lag $\ell = 0, \ldots T-1$. This implies x_t is stationary conditional on $\widehat{\gamma}_\ell \in (-\infty, \infty)$ for $\ell = 0, \ldots, T-1$.

The sample periodogram is computed in one of two ways. The transfer function of a stationary ARMA model can be engaged to construct the ACGF. Suppose x_t is a ARMA(p, q), $\theta(\mathbf{L})x_t = \phi(\mathbf{L})u_t$, where $\theta(\mathbf{L}) = 1 - \sum_{j=1}^{p}\theta_j$, $\phi(\mathbf{L}) = 1 + \sum_{k=1}^{q}\phi_k$, and $u_t \sim \mathcal{N}(0, \sigma_u^2)$. Assuming $\theta(\mathbf{L})$ and $\phi(\mathbf{L})$ are invertible, the estimated population specturm is

$$\widehat{S}_x(\omega) = \frac{\widehat{\sigma}_u^2}{2\pi}\left[\frac{\left(1 + \sum_{k=1}^{q}\widehat{\phi}_k e^{-i\omega k}\right)\left(1 + \sum_{k=1}^{q}\widehat{\phi}_k e^{i\omega k}\right)}{\left(1 - \sum_{j=1}^{p}\widehat{\theta}_j e^{-i\omega j}\right)\left(1 - \sum_{j=1}^{p}\widehat{\theta}_j e^{i\omega j}\right)}\right],$$

where $\widehat{\phi}_k$, $\widehat{\theta}_j$, and $\widehat{\sigma}_u^2$ represent estimates of the coefficients of the ARMA(p, q) of x_t; see

Akaike (1969), Parzen (1974), and Hamilton (1994, pp. 154–155, 164–165) and Kano and Nason (2014) for an example using a structural VAR.

The other estimator of the sample periodogram is non-parametric. The non-parametric estimator of the population spectrum and ACGF is built on the fast Fourier transform (FFT). The FFT uses the discrete Fourier transform to calculate the ACGF as $f(\omega) = \sum_{\ell=-\infty}^{\infty} \gamma_\ell e^{-i\omega\ell}$. The Fourier inversion theorem recovers the ACGF using $\gamma_\ell = 0.5\pi^{-1}\int_{-\pi}^{\pi} f(\omega)e^{-i\omega\ell}d\omega$. As a result, the FFT is a computational efficient algorithm to compute the sample periodogram.

Computing a non-parametric sample periodogram also assumes there is some frequency φ that when close to ω (*i.e.*, say, within a $\epsilon > 0$), the same holds for $S_x(\varphi)$ and $S_x(\omega)$. This suggests weighting or smoothing $\widehat{S}_x(\omega)$ to satisfy this result. The smoothing function, which is also know as a window, is $w_\ell = w\left(\omega_{\ell+m}, \omega_\ell\right)$. Plugged into the sample periodogram it produces the smoothed spectrum

$$\widehat{S}_x(\omega) = \frac{1}{2\pi}\sum_{\ell=-T+1}^{T-1} \widehat{\gamma}_\ell e^{-i\omega\ell} = \frac{1}{2\pi}\left[\widehat{\gamma}_0 + 2\sum_{\ell=1}^{T-1} w_\ell \widehat{\gamma}_\ell \cos\left(\omega\ell\right)\right], \tag{2.9}$$

where $\hbar$ is the bandwidth of w_ℓ, $\sum_{\ell=1}^{\hbar} w_\ell = 1$, the period of the cycle is $2\pi/\omega$ for a cycle of ω, $\omega_\ell = 2\pi\ell/T$ for a sample of length T, and $T/\ell = 2\pi\ell/\omega_\ell$ defines the associated period.

There is a long menu of window functions from which to choose. The Bartlett and Bartlett-Hanning windows are popular choices for w_ℓ to compute the smoothed spectral density (2.9). The former window function sets the unnormalized weights to $\mathfrak{w}_{BR,\ell} = 1 - q/(\hbar - 1)$ for $\ell = 1, \ldots, \hbar{-}1$ and $\mathfrak{w}_{BR,\ell} = 0$ for $\ell \geq \hbar$. The latter window function produces the unnormalized weights $\mathfrak{w}_{BH,\ell} = 0.62 - 0.48\left|\ell_{\hbar} - 0.5\right| + 0.38\cos\left[2\pi\left(\ell_{\hbar} - 0.5\right)\right]$, where $\ell_{\hbar} = \ell/(\hbar - 1)$ for $\ell = 1, \ldots, \hbar{-}1$ and $\mathfrak{w}_{BH,\ell} = 0$ for $\ell \geq \hbar$. Both windows require $\hbar > 3$ (and its is an odd integer). The normalized weights are $w_{k,\ell} = \mathfrak{w}_{k,\ell}\Big/\sum_{j=1}^{\hbar}\mathfrak{w}_{k,j}$, where $k = BR, BH$.

Otherwise, there is no theory for fixing $\hbar$. Applying a window to smooth the sample periodogram shrinks its variance, but introduces bias into the smoothed spectral density (2.9).

This is a general result for choosing window widths in nonparametric settings. There is a trade off between a smaller variance and the potential for creating bias. This is the case because, although the sample periodogram is an asymptotically unbiased estimator of the population spectrum, the sample periodogram has substantial variance. There are methods to choose $\hbar$ optimally that aim to trade off variance for bias, but for most purposes in macroeconometrics the payoff of these methods is less than the costs.

Figure 3 depicts the normalized weights of the Bartlett and Bartlett-Hanning window functions for $\hbar = 7$ and 21. The solid lines plot $w_{BR,\ell}$ while the dot-dashed lines denote $w_{BH,\ell}$. Note the indexes of the weights have been centered around zero. The Bartlett window is a triangle or tent function. Increasing $\hbar$ shrinks $w_{BR,\ell}$ because the weights are spread across a larger range. However, the characteristic shape of the Bartlett window function is left unchanged. The Bartlett-Hanning window has fatter tails and is more peaked around zero compared with the Bartlett window function. At $\hbar = 21$, a bell shape begins to take shape in the Bartlett-Hanning window function. This is its characteristic shape because $w_{BH,\ell} > w_{BR,\ell}$ at small ℓ while the inequality reverses as ℓ increases.

Figure 4 plots the smoothed spectral densities of U.S. output growth and its first difference. The smoothing function is the Bartlett-Hanning window with $\hbar = 11$. The top panel of figure 4 contains the spectral densities of quarter over quarter and year over year growth in per capita real GDP, $\widehat{S}_{\Delta y}$ and $\widehat{S}_{(1-\mathbf{L}^4)y}$, from 1948Q1 to 2019Q4. These plots show the explanatory power of the autocovariances of $\Delta \ln y_t$ and $(1 - \mathbf{L}^4) \ln y_t$ are concentrated between eight and 32 quarters per cycle (*i.e.*, peaks in $\widehat{S}_{\Delta y}$ and $\widehat{S}_{(1-\mathbf{L}^4)y}$ occur in these frequencies). For example, the peak or maximum power of $\widehat{S}_{\Delta y}$ occurs at nearly 12 quarters per cycle. The spectral density of $(1 - \mathbf{L})^2 \ln y_t$ has its greatest power in the short-run frequencies higher than six quarters per cycle as displayed in the bottom panel of figure 4. This plot is consistent with the idea that over differencing a stationary time series eliminates fluctuations at the business cycle and lower frequencies. This also suggests strong mean reversion in $(1 - \mathbf{L})^2 \ln y_t$.

Plots of the smoothed spectral densities of chained weighted GDP deflator inflation and $(1-\mathbf{L})^2 \ln P_t$ appear in figure 5. The construction and sample period of $\widehat{S}_{\Delta P}$, $\widehat{S}_{(1-\mathbf{L}^4)P}$, and $\widehat{S}_{(1-\mathbf{L})^2P}$ parallel the discussion of the previous paragraph. The bottom panel of figure 5 shows $\widehat{S}_{(1-\mathbf{L})^2P}$ has greatest power in the short-run. This is also true for $\widehat{S}_{(1-\mathbf{L})^2Y}$ in figure 4. However, the spectral densities of $\Delta \ln P_t$ and $(1-\mathbf{L}^4) \ln P_t$ have maximum power at frequency zero (*i.e*, the infinite long-run). There are smaller peaks in $\widehat{S}_{\Delta P}$ and $\widehat{S}_{(1-\mathbf{L}^4)P}$ at nine years per cycle, which is a lower frequency than often attributed to the business cycle. A third and smaller peak is in the business cycle frequencies at about 15 quarters per cycle. Nonetheless, the power explaining fluctuations in inflation is concentrated in the growth to long-run frequencies.

2.b.ii *Wiener-Kolmogorov filters*

One class of linear filters takes the existence of an *ideal* filter as given. An ideal filter is

$$x_t = \sum_{\ell=-\infty}^{\infty} C_\ell \xi_{t+\ell}, \tag{2.10}$$

where x_t and ξ_t are a stationary time series of infinite length, $\sum_{\ell=-\infty}^{\infty} C_\ell = C(\mathbf{L})$, is the impulse response function (IRF) of this ideal filter, and $\sum_{\ell=-\infty}^{\infty} |C_\ell| < \infty$ (*i.e.*, absolutely summable). The stationarity of ξ_t is useful, but is not necessary. The ideal linear filter (2.10) is also an infinite-order moving average, MA(∞). The MA(∞) is a linear mapping from ξ_t to x_t.

A finite-order IRF minimizes the MSE between the ideal filter (2.10) and its finite sample approximation. The approximation is

$$\hat{x}_t = \sum_{\ell=-s}^{k} \widehat{C}_{t,\ell} \xi_{t+\ell}, \tag{2.11}$$

where $\{\hat{x}_t\}_{t=1}^{T}$ is the finite sample cousin of $\{x_t\}_{t=-\infty}^{\infty}$ and s and k define a window of $s+1+k$ observations on which the finite sample filter is computed at date t. The minimization problem solves for the finite-order IRF

$$\left\{\widehat{C}_{t,\ell}\right\} = \arg\min \mathbf{E}\left\{(x_t - \hat{x}_t)^2\right\} \tag{2.12}$$

which produces a Wiener-Kolmogorov filter, where $k = s$ is not necessary (*i.e.*, the approximation does not impose symmetry, $\hat{C}_{t,\ell} \neq \hat{C}_{t,-\ell}$). The finite sample Wiener-Kolmogorov filter (2.12) of the ideal filter (2.10) also does not restrict the IRF to be time-invariant.

The mapping from the ideal filter (2.10) to the finite sample Wiener-Kolmogorov filter depends on the transfer function. The transfer function of the ideal linear filter (2.10) is $C\left(e^{-i\omega}\right)$. A useful way to factor the transfer function is $C\left(e^{-i\omega}\right) = \mathcal{G}a\left(\omega\right) e^{-i\mathcal{P}h\omega}$, where the gain of the filter, $\mathcal{G}a\left(\omega\right) = \left|C\left(e^{-i\omega}\right)\right|$, is the modulus of $C\left(\omega\right)$, $\left|C\left(e^{-i\omega}\right)\right|$, $\mathcal{P}h = \tan^{-1}\left(-C_{imag}\left(\omega\right) / C_{real}\left(\omega\right)\right)$, measures the phase shift (*i.e.*, change in the lead-lag behavior of the ACGF) induced by the filter, and its decomposition into real and imaginary parts, $C\left(\omega\right) = C_{real}\left(\omega\right) + iC_{imag}\left(\omega\right)$. The inverse Fourier theorem states the coefficients of the IRF are recovered from $C_{t,\ell} = 0.5\pi^{-1}\int_{-\pi}^{\pi} C_t\left(e^{-i\omega}\right) e^{i\omega} d\omega$. Since the IRF of the ideal filter is generated in the frequency domain, Schleicher (2004) suggests using

$$\hat{C}_{t,\ell} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|C_t\left(e^{-i\omega}\right) - \hat{C}_t\left(e^{-i\omega}\right)\right|^2 d\omega,$$

to recover the IRF of the finite sample Wiener-Kolmogorov filter.

The transfer function solves the problem common to several linear filters used by macroeconomists that the IRFs of these filters are analytic only in the frequency domain. Baxter and King (1999) develop a filter that is a leading example. Their filter rests on the widely held notion that the business cycle lives in frequencies between eight (two) and 32 (eight) cycles per quarter (year). Restricting the IRFs, $\left\{\hat{C}_t\right\}$, on these frequencies, defines a band pass filter because the Baxter-King filter removes frequencies lower than eight years per cycle and higher than two years per cycle. Furthermore, the Baxter-King filter aims to leave the lead-lag structure of a time series unchanged (*i.e.*, no phase shifts), remove trend components in an integrated time series resulting in a stationary business cycle component, and approximate the ideal band pass filter (*i.e.*, annihilate all fluctuations above and below the cutoff frequencies).

Nonetheless, Murray (2003) critiques the Baxter-King filter, given x_t satisfies the BN decomposition. The Baxter-King filter can generate spurious cyclical behavior when $x_t \sim I(1)$.

Harvey and Trimbur (2003) show their local level trend model of equations (2.8), (2.7), and the associated system of stochastic cycles can duplicate the Baxter-King filter by changing the order of integration of x_t.

2.b.iii *Optimal Linear Filters*

The transfer functions of the ideal linear filter (2.10) and its finite sample approximation (2.11) are used by Schleicher (2004) to solve the mininization problem (2.12). His solution starts with

$$\mathbf{E}\left\{(x_t - \hat{x}_t)^2\right\} = \int_{-\pi}^{\pi} \left| C\left(e^{-i\omega}\right) - \hat{C}_t\left(e^{-i\omega}\right)\right|^2 f_x(\omega)d\omega,$$

which is a restatement of the transfer function approximation of the ideal filter. Next, Schleicher (2004) employs the ARMA(p, q) of x_t is $\theta(\mathbf{L})x_t = \phi(\mathbf{L})u_t$, where the lag polynominals $\theta(\mathbf{L})$ and $\phi(\mathbf{L})$ are invertible. The ARMA can be written as an MA(∞), $x_t = \left[\phi(\mathbf{L})\big/\theta(\mathbf{L})\right]u_t = \psi(\mathbf{L})u_t$, the coefficients of $\psi(\mathbf{L})$ are absolutely summable (while the restriction on u_t can be relaxed to $\mathcal{WN}$ instead of Gaussian). These assumptions let Schleicher (2004) recast the mininization problem (2.12) as

$$\mathrm{Min}_{\hat{C}_{t,\ell}} \mathrm{MSE}\left(\hat{C}_{t,\ell}\right) = \int_{-\pi}^{\pi} \left| C\left(e^{-i\omega}\right) - \hat{C}_t\left(e^{-i\omega}\right)\right|^2 f_x(\omega)d\omega. \tag{2.13}$$

The optimal choice of $\left\{\hat{C}_t\right\}$ weights the distance between the infinite length transfer function and its finite sample approximation by the spectral density of x_t frequency by frequency.

Schleicher (2004) extends this result to an arbitrary order of integration for $x_t \sim I(M)$ by invoking two assumptions. First, the stronger restriction of square summability is placed on $\psi(\mathbf{L})$, $\sum_{j=1}^{\infty} \psi_j^2 < \infty$. Next, the transfer function of any candidate finite sample filter, $\left|\hat{C}\left(e^{-i\omega}\right)\right|$, equals the transfer function of the ideal filter, $\left|C\left(e^{-i\omega}\right)\right|$, at frequency zero. In this case, the solution $\left\{\hat{C}_t\right\}$ to the minimization problem (2.13) renders its left hand side finite. However, Schleicher (2004) suggests constructing $\left\{\hat{C}_t\right\}_{t=1}^{T}$ parametrically by estimating a ARMA(p, q) after differencing x_t to render it stationary. Hence, his procedure computes the optimal coefficients of the linear filter conditional on a ARMA(p, q) for x_t or a stationary transformation of it. These results are in the spirit of the BN smoother created by Proietti and Harvey (2000).

2.b.iv *The Hodrick and Prescott (1997) Linear Filter*

There is a way to construct an optimal linear filter without having to parameterize a ARMA. A leading example of this approach is the linear filter of Hodrick and Prescott (1997). They start with the decomposition (2.1) of $\ln y_t$ into trend, τ_t, and cycle, ε_t. Minimizing the square of ε_t subject to a constraint on the squared second difference of τ_t yields the quadratic program

$$\mathbf{Min}_{\{\tau_t\}_{t=1}^{T}} \sum_{t=1}^{T} \left[\left(\ln y_t - \tau_t \right)^2 + \lambda \left[\left(\tau_t - \tau_{t-1} \right) - \left(\tau_{t-1} - \tau_{t-2} \right) \right]^2 \right],$$

where the Lagrange multiplier, λ, on the constraint is often referred to as the smoothing parameter of the Hodrick-Prescott filter and the constraint is the second different of τ_t.

Movements in τ_t are penalized by the constraint of the Hodrick-Prescott minimization problem. The penalty function is an incentive to smooth τ_t because its second difference places costs on low frequency fluctuations. This motivates calling λ the Hodrick-Prescott smoothing parameter. As $\lambda \to \infty$ (given T), the incentive to smooth τ_t becomes large enough to transform it into a deterministic function of time; see Cogley and Nason (1995a) and Hamilton (2018).

The Hodrick-Prescott filter has two more well known features. Singleton (1988) shows the Hodrick-Prescott filter passes frequencies higher than 32 quarters per cycle when $\ln y_t \sim I(0)$. In this case, the Hodrick-Prescott filter works as a high pass filter. Second, King and Rebelo (1993) present analysis that the Hodrick-Prescott filter renders stationary $\ln y_t \sim I(M)$ for M = 1, 2, 3, 4. Pollock (2000) and Gomez (2001) also point out the Hodrick-Prescott filter is in the class of Butterworth filters used in electrical engineering. However, Phillips and Jin (2021) note there other traditions in linear filtering that can lay claim to the Hodrick-Prescott filter.

There is no analytic solution in the time domain for the Hodrick-Prescott filter. Cogley and Nason (1995a) build on King and Rebelo (1993) to obtain a solution to the (infinite length)

Hodrick-Prescott minimization problem that is the time-invariant two-sided linear filter $\hat{\varepsilon}_t = \mathcal{HP}(\mathbf{L}) \ln y_t$, where $\mathcal{HP}(\mathbf{L}) = \frac{|\varrho|^2}{\mathbf{L}^2}\left[\frac{1 - 2\varrho_{real}\mathbf{L} + |\varrho|^2\mathbf{L}^2}{1 - 2\varrho_{real}\mathbf{L}^{-1} + |\varrho|^2\mathbf{L}^{-2}}\right](1-\mathbf{L})^4$, ϱ^{-1} is the stable root of the lag polynomial $\lambda^{-1}\mathbf{L}^2 + (1-\mathbf{L})^4 = 0$,and $|\varrho|$ is the modulus of ϱ. This solution is explicit about the King and Rebelo (1993) observation that the Hodrick-Prescott filter can make stationary any time series with up to four unit roots. Hamilton (2018) offers a similar two-sided solution, $\hat{\tau}_t = \mathcal{F}^{-1}(\mathbf{L}) \ln y_t$, but for the finite sample Hodrick-Prescott minimization problem, where $\mathcal{F}(\mathbf{L}) = 1 + \lambda(1-\mathbf{L}^{-1})^2(1-\mathbf{L})^2$ and $\mathcal{F}^{-1}(\mathbf{L}) = \frac{\lambda^{-1}\mathbf{L}^2}{\lambda^{-1}\mathbf{L}^2 + (1-\mathbf{L})^4}$. Hence, the Cogley and Nason (1995a) and Hamilton (2018) solutions involve lag polynomial functions implying transfer functions with infinite order IRFs. This explains the focus on computing exact (numerical) expressions to plug in for $\mathcal{F}^{-1}(\mathbf{L})$ by McElroy (2008), de Jong and Sakarya (2016), and Cornea-Madeira (2017).

Hodrick and Prescott (1997) recognized the problem, which lead them to use the Kalman filter to generate τ_t and ε_t. However, initial conditions for the trend, τ_0 and τ_{-1} are needed to run the Kalman filter because $\mathbf{L}^2$ appears in the numerator of $\mathcal{F}^{-1}(\mathbf{L})$. As Hamilton (2018) among others discuss, standard practice is to endow τ_0 and τ_{-1} with a covariance matrix that has a diffuse prior (*i.e.*, large determinate) to initialize the Kalman filter. However, this prior has an interesting effect on the Hodrick-Prescott filter that is a source of debate.

Advocates of the Hodrick-Prescott filter often are interested in a DGP in which $(1-\mathbf{L})\tau_t$ is a random walk and $\varepsilon_t \sim \mathcal{WN}$. Harvey and Jaeger (1993), Gomez (1999), and proposition 1 of Hamilton (2018) show the same DGP is created by restricting $\zeta_{1,t} = \vartheta_t = 0$ in equations (2.4) and (2.5) of the Harvey and Jaeger (1993) local level trend UC model. If $\lambda = \sigma^2_{\varpi}\big/\sigma^2_{\zeta,2}$, the Hodrick-Prescott filter and the restricted local level trend UC model produce identical estimates of trend and cycle. The observational equivalence of the Hodrick-Prescott DGP implicitly assumes the innovation to $(1-\mathbf{L})^2\tau_t$, ε_t, and (τ_0, τ_{-1}) are uncorrelated. Moreover, a DGP built on white noise transitory component seems to be at odds with a priori notions of business cycles in developed economies.

This begs the question of the standard calibration of the Hodrick-Prescott smoothing parameter for quarterly data, which sets $\lambda = 1600$. This calibration is grounded in the Hodrick and Prescott (1997) argument that the standard deviations of ε_t and $(1 - \mathbf{L})^2\tau_t$ are 5% and 0.125%, respectively. In their view, these values represent "moderately large" volatility in trend and cycle, which seem to imply values greater than the sample means of σ_ε and $\sigma_{\Delta^2\tau}$. The upshot is $\sqrt{\lambda} = 40 = 5/0.125$. Hence, the Hodrick-Prescott filter generates volatility in its cycle that is orders of magnitude greater than volatility in the trend in quarterly data. Nonetheless, de Jong and Sakarya (2016) are clear there is no consensus about calibrating λ when applying the Hodrick-Prescott filter to data at the monthly and annual frequencies. Perhaps, the reason is Phillips and Jin (2021) show the choice of λ cannot be divorced from the sample size T.

Issues with the calibration of λ motivates Hamilton (2018) to estimate the restricted local level trend UC by maximum likelihood. He reports estimates of λ ranging from less than 0.025 to nearly 10 for macro aggregates, prices, and asset prices and returns. The non-optimality of $\lambda = 1600$ suggests the potential for the Hodrick-Prescott filter to produce spurious trends and cycles using this calibration as argued by Harvey and Jaeger (1993), Cogley and Nason (1995a), and Hamilton (2018). Spuriousness occurs when the filter produces trend and cycle that do not replicate the true underlying DGP.

Two more critiques of the Hodrick-Prescott filter are also about the ways it can induce spurious behavior in detrended data. The Hodrick-Prescott filter is known for creating different behavior at the beginning and end of the sample compared with observations in the middle, where the length of the start up and wind down is a function of T; see Nelson and Kang (1981). The cycle can exhibit unit root behavior early and late in the sample. A point made by de Jong and Sakarya (2016), Hamilton (2018), and Phillips and Jin (2021). Schleicher (2004) proposes a solution to the problem.

The Hodrick-Prescott filter produce estimates consistent with $\varepsilon_t \sim I(0)$ in the middle of the sample. Nonetheless, Hamilton uses results from Cogley and Nason (1995a) about applying the Hodrick-Prescott filter to $I(1)$ time series to show it creates spurious periodicity in the middle of the sample. When $\ln y_t \sim I(1)$, the underlying DGP has a BN trend. The essence of the problem is $\mathcal{HP}(\mathbf{L})$ has a fourth difference operator, which after differencing $\ln y_t$ leaves three roots unattended, $\varepsilon_t = \mathcal{HP}(\mathbf{L})(1-\mathbf{L})^3\Delta \ln y_t = \lambda\mathcal{F}^{-1}(\mathbf{L})(1-\mathbf{L})^3\mathbf{L}^{-2}\Delta \ln y_t$. Over differencing $\Delta \ln y_t$ and hitting it with $\mathcal{HP}(\mathbf{L})$ or $\mathcal{F}^{-1}(\mathbf{L})$ is a source of spurious periodicity in τ_t and ε_t.

Hodrick (2020) defends the Hodrick-Prescott filter in two parts. First, he argues Hodrick and Prescott (1997) and other advocates of their filter never put weight on a DGP in which the cyclical component is white noise. Rather, Hodrick and Prescott (1997) were arguing for a nonparamteric model of trend and cycle because, as Hodrick (2020) notes, they claimed that specifying a stochastic process for τ_t alone is insufficient to identify the true DGP of sample data. Second, he estimates widely used ARMA and UC models by ML to calibrate simulation experiments that produce Hodrick-Prescott filtered cycles of aggregate U.S. data. The simulated ARMA models are responsible for Hodrick-Prescott filtered cycles that display spurious periodicity. This is not the case for simulated Hodrick-Prescott filtered cycles generated by the UC models. Since the Hodrick-Prescott filter replicates the true underlying cyclical dynamics of sophisticated DGPs that mimic aggregate fluctuation in developed economies, Hodrick (2020) contends this is evidence supporting the use of the Hodrick-Prescott filter in macro research.

2.c *Least Squares Filtering*

Hamilton (2018) develops a method employing least squares regressions to decompose aggregate variables into trend and cycle. The approach is grounded in three facts about $\ln y_t \sim I(M)$ for $M > 1$. The first fact is $(1-\mathbf{L})^M \ln y_t \sim I(0)$. Hence, the least squares regression

$$\ln y_{t+h} = \alpha_{1,h} \ln y_t + \sum_{k=2}^{M} \alpha_{k,h}(1-\mathbf{L})^{j-1} \ln y_t + e_{h,t},$$

yields estimates of the coefficients on the regressors that sum to one and regression errors,

$\{e_{h,t}\}_{t=h+1}^{T}$, that are stationary asymptotically. The asymptotic theory relies on arguments found in section 5 of Sims, Stock, and Watson (1990). Their theory also can be used to show similar asymptotic results hold for the regression $\ln y_{t+h} = \sum_{k=1}^{M} a_k \ln y_{t-k} + e_{h,t}$ that unwinds the differences $(1-\mathbf{L})\ln y_{t-1}$, $(1-\mathbf{L})^2 \ln y_t, \ldots, (1-\mathbf{L})^{d-1} \ln y_t$ and rearranges the regressors, where the a_ks are sums of $\alpha_{k,h}$, $k = 1, \ldots, m$. This regression is

$$\ln y_{t+h} \;=\; \beta_0 \;+\; \sum_{j=1}^{p} \beta_j \ln y_{t-j+1} \;+\; \varsigma_{t+h}, \tag{2.14}$$

in sample. Hamilton (2018) estimates it by OLS, where $\{\hat{\varsigma}_{t+h}\}_{t=h+1}^{T}$ is the estimated cycle.

The parameters h and p must be calibrated to implement the Hamilton detrending regression (2.14). Hamilton (2018) recommends $h = 8$ and $p = 4$ for quarterly data. Setting $h = 8$ depends on "the standard benchmark" of a two year business cycle horizon. The choice of p is to insure stationarity of the regression error ς_{t+h}. Since proposition 4 of Hamilton (2018) restricts $M \leq p$, the goal is to set p large enough to render $(1-\mathbf{L})^M \ln y_t \sim I(0)$ in large sample. This suggests intuition for $p = 4$ is motivated by the King and Rebelo (1993) result that the Hodrick-Precott filter renders stationary $\ln y_t \sim I(M)$ for $M \leq 4$.

Hodrick (2020) critiques the Hamilton (2018) detrending regression (2.14). The critique involves comparing the Hamilton trend, $\mathbf{E}\left\{\ln y_{t+h} \,\middle|\, \ln y_t, \ldots, \ln y_{t-p+1}\right\}$, with the BN trend, $\lim_{h\to\infty,\, p\to\infty} \mathbf{E}\left\{\ln y_{t+h} \,\middle|\, \ln y_t, \ldots, \ln y_{t-p+1}\right\}$. The double limit eliminates correlation between the information set and innovation to the BN trend. The same holds for the Hamilton trend, but Hodrick (2018) notes the BN trend relies on a larger information set, $\ln y_t, \ldots, \ln y_{-\infty}$, relative to the Hamilton regression conditioning information, $\ln y_t, \ldots, \ln y_{t-p+1}$. A consequence is the Hamilton detrending regression (2.14) may fail to use all the information available to estimate the Hamilton cycle, $\varsigma_{t+h} = \ln y_{t+h} - \mathbf{E}\left\{\ln y_{t+h} \,\middle|\, \ln y_t, \ldots, \ln y_{t-p+1}\right\}$. This is a large sample problem pointing to misspecification of the Hamilton detrending regression (2.14). If this regression does not condition on all available information, there is also potential for small sample bias in the OLS estimator. These large and small sample problems are unstudied at least at this moment. Schüler (2021) finds the Hamilton detrending regression alters the lead-lag properties of a time series frequency by frequency (*i.e.*, a phase shift). Moreover, the impact differs for different times series.

2.d *Summing Up*

This section began with a question that remains open. The question is whether to prefer UC models, linear filters, or least squares filtering for constructing sample moments of the business cycle to test DSGE models. The UC models and least squares filtering, which are parametric methods, can suffer from large sample problem of misspecification and biased estimates in small sample. Linear filters are argued to avoid these problems because these are nonparametric estimators of trend and cycle. However, linear filtering is subject to ad hoc calibration of high and low pass frequency cutoffs and window width and smoothing parameters, which can induce spurious periodicity. The bottom line is there seems to be no clear choice along the time series econometric dimension.

Figures 5, 6, and 7 reinforce this point. Plots of Hodrick-Prescott, Baxter-King, and Hamilton cycles (*i.e.*, gaps) appear in the top, middle, and bottom panels of these figures. Output gaps, price level gaps, and inflation gaps are found in figures 5, 6, and 7, respectively. Output is per capita chain weighted real GDP. The price level is the chain weighted GDP deflator, which is the source of inflation gaps plotted in figure 7. The quarterly sample runs from $1948Q1$ to $2019Q4$. However, the Baxter-King depends on a smoothing window of width $\hbar$ that removes $\hbar$ observations at the beginning and end of the sample while h observations are eliminated at the start of the sample to estimate Hamilton cycles. The figures also contain NBER recession dates as silver vertical shading.

The Hodrick-Prescott, Baxter-King, and Hamilton output gaps display business cycle fluctuations consistent with the NBER recession dates in figure 5. There are troughs in these output gaps around the time the NBER dating committee places the end of a recession as shown in the bottom panel of figure 5. These plots also reveal the Hamilton output gap is more volatile compared with the Hodrick-Prescott and Baxter-King filters.

The same is often not true for plots of the price level gaps in figure 6. The bottom panel of the figure contain Hodrick-Prescott and Baxter-King price level gaps that peak during NBER dated recessions, especially pre-1984, but the Hamilton price level gap lacks this co-movement with the NBER recession dates during the entire sample. After 1983, there is a noticeable drop in the magnitude of the fluctuations in the three price level gaps.

Figure 7 depicts inflation gaps that behave similar to the output gaps of figure 5. Gap inflation troughs at the end of NBER dated recessions while peaking between the vertical silver shading, which indicates gap inflation is procyclical. There is also a substantial drop in the volatility of the Hodrick-Prescott and Baxter-King inflation gaps post-1983, but the Hamilton inflation gap does not display nearly as sizable reduction.

Three more features of the Hodrick-Prescott, Baxter-King, and Hamilton output gaps, price level gaps, and inflation gaps are worth mentioning. First, the top panel of figures 5 and 7 are clear that as λ becomes large the Hodrick-Prescott output and inflation gaps fall onto linearly detrended output and inflation. For the Hodrick-Prescott price level gap and large λ, the comparison is to the quadratically detrended price level as in the top panel of figure 6 because stationarity of $\Delta\pi_t$ implies $\ln P_t \sim I(2)$. Next, Baxter-King output and inflation gaps differ most for a small window width of $\kappa = 4$ quarters in the middle panel of figures 5 and 7. As κ increases to 12 and 32 quarters, these differences are less apparent. The middle panel of figure 5 shows the Baxter-King price level gap is quantitatively different across the three values of κ. Third, the bottom panels of figures 5, 6, and 7 display Hodrick-Prescott and Baxter-King output, price level, and inflation gaps that are difficult to distinguish on the ocular metric. Plots of the Hamiltion gaps in the same panels are easy to tell apart from the Hodrick-Prescott and Baxter-King gaps. Canova (1998a) and Hodrick (2020) have also noted this.

Canova (1998a, 1998b) and Burnside (1998) debate the use of different detrending methods and construction of "business cycle facts." They agree that relying on a single detrending method leads to incorrect inferences about the ability of DSGE models to match sample moments of the business cycle. Different detrending methods can uncover different of stories about the business cycle. Hence, the issue is not so much that one detrending method dominates another. Instead, their disagreement is about the methods employed to select the most powerful moments on which to assess the match between macro theories and sample data and the econometric methods employed to conduct the evaluation.

Perhaps, there is a way to move beyond this debate. For example, the discussion in Hansen and Sargent (1993) suggests that cross-frequency restrictions matter for decomposing aggregate time series into trend and cycle. Cross-frequency restrictions are about co-movement, say, in the trend and cycle of aggregate output. Morley, Nelson, and Zivot (2003) make this point by estimating the correlation of the innovations to τ and ε_t, $\varrho_{\eta,\upsilon}$, in their UC model. Only when they estimate $|\varrho_{\eta,\upsilon}| < 1$ does the UC model produce a BN trend that dominates fluctuations in U.S. output. Similarly, Gregory and Smith (1996) report that the evaluation of RBC theory is sensitive to assumptions about the DGP of the trend in the sample data. Kulish and Pagan (2019) make a related point, but instead focus on the persistence of the cyclical component of aggregate data.

Another approach to the Canova-Burnside debate is to study the impact different filters have on tests of DSGE models. As the debate reveals, proponents of the Hodrick-Prescott filter have never accepted this as a legitimate criticism. Nonetheless, the Canova-Burnside debate is about whether some methods of separating trend from cycle may reduce the power of tests of business cycle theory. The notion of power is the effect the procedure that decomposes trend and cycle has on the ability of an econometrician to distinguish between competing DSGE models conditional on moments of the sample data. A way to avoid this problem is to construct tests that are invariant to linear filters. Gregory and Smith (1990, 1991, 1996), Sims (1996), and Kehoe (2007) advocate this approach. It is implemented by Cogley and Nason (1993, 1995a), Nason and Cogley (1994), Nason and Rogers (2006), Canova and Ferroni (2011), and Kano and Nason (2014), among others. The next section studies a canonical RBC model as a prelude to testing DSGE models using methods invariant to linear filtering.

3. Calibrating, Linearizing, and Solving a RBC Model

The idea of RBC theory dates from the time-to-build paper by Kydland and Prescott (1982). However, the notion of imposing a rational expectations equilibrium (REE) concept on a stochastic growth model to study business cycle fluctuations can be traced to Brock (1974). From the perspective of the history of macroeconomics, the explosion of RBC papers in the latter half of the 1980s and first half of the 1990s is attributed to an idea at the right place at the right time.

The appeal of RBC theory is the way in which it connects theory to data. RBC theory explains business cycle fluctuations by propagating exogenous shocks through the interaction of the primitives of an economy, which are, for example, preferences, technology, and market structure. This is not a new idea. Slutsky (1937) shows that summing sequences of white noise random deviates (*i.e.*, innovations to productivity shocks) generates serially correlated time series. Frisch (1933) develops the notion that white noise shocks can generate impulse response dynamics given propagation mechanisms grounded in economic theory. A strength of RBC theory is that it is explicit about the way the business cycle propagation mechanism operates. This allows for a thorough examination of its ability to match moments of the data.

This raises the issue of which data to match. Initially, the RBC literature stressed the ratio of standard deviations and contemporaneous correlations between output, consumption, investment, labor input (either hours worked, employment, or some combination of the two), and average labor productivity (the real wage in a one-sector growth model), but Kydland and Prescott (1982) is a notable exception. Hence, RBC theory emphasized the relative volatility and contemporaneous co-movement aspects of business cycle fluctuations. Later waves of research took the dynamics of RBC models seriously. This work confronts RBC theory directly with observed business cycle persistence and dynamic co-movement.

To summarize, RBC theory consists of set of methods and tools to

(i) construct dynamic stochastic general equilibrium (DSGE) models to compute theoretical business cycle moments,

(ii) measures sample business cycle fluctuations identical to the way theoretical moments are, and

(iii) tests theories against the sample data.

This summary suggests there is an econometric interpretation of RBC theory. The econometric interpretation views DSGE models as restricted DGPs. A DSGE model places theoretical (and possibly arbitrary) restrictions on the multivariate probability distribution process that generates, say, output, consumption, investment, and employment. The restrictions use economic theory to organize the information and data the real world offers to evaluate DSGE models.

Quite often proponents of RBC theory will make the statement, "*All models are false. But it takes a model to beat a model.*" Since a RBC model is an abstraction of the aggregate economy, along some dimension the data will reject the model. That is not a powerful statement. A powerful statement, as well as statement of science, is that when the data rejects the RBC model, something is discovered about the data and RBC theory. The scientific process involves making this discovery the motivation for the next model to be studied. Hence, tests of RBC theory can be thought of specification searches across classes of falsifiable models to learn about the sources, causes, and propagation mechanisms of business cycles. Rather than an attempt to make meta-statements about the sources and causes of business cycles, RBC theory is a process of learning about the ability of DSGE models to match moments in the data.

3.a *A Canonical RBC Model*

Kydland and Prescott (1982) put together a RBC model that contains time to build in capital accumulation, habit formation in leisure, and signal extraction in the fundamental shocks. A reason that Kydland and Prescott becomes so important is that they claim their DSGE model mimics much of the behavior of the U.S. business cycle. This surprises some economists, especially Keynesian and New Keynesian economists. The moments of the U.S. business cycle Kydland and Prescott cannot capture are found in the labor market. For example, Kydland and Prescott's time to build model fails to match the volatility of employment (relative to output) over the cycle and the contemporaneous co-movement of average labor product and employment. Much of the first wave of RBC literature is concerned with matching these labor market moments over the business cycle.

Rather than work with this complex structure, a much simpler RBC model is constructed below. It is a single consumption good one-sector growth model that includes final goods and household sectors and a government. The goal is to study the business cycle properties of the model. The optimality conditions of the RBC model place restrictions on the equilibrium path of the economy. Study of the implications of the restrictions for moments of the business cycle requires specification of the economic primitives, which are preferences and technology in this RBC model (because all markets are assumed perfectly competitive). Once utility and production functions have been chosen, parameters values have to be selected. This is the process of calibration and is connected to computation of the steady state. With the RBC model calibrated and the steady state defined, a numerical solution can be computed.

3.a.i *The Final Goods Sector*

The final goods sector consists of many identical firms that have access to a constant returns to scale (CRS) technology in capital, K_t, and labor, N_t, driven by labor augmenting technical change A_t common to all firms. Output, Y_t, is produced with this technology $Y_t = \mathcal{F}\left(K_t, A_tN_t\right)$. For the moment, treat A_t as an exogenous, but not yet specified total factor productivity (TFP) shock. The TFP shock is observed at the beginning of date t. The CRS production technology has standard restrictions, which are $\mathcal{F}_K(\cdot,\cdot) > 0$, $\mathcal{F}_{KK}(\cdot,\cdot) < 0$, $\mathcal{F}_N(\cdot,\cdot) > 0$, $\mathcal{F}_{NN}(\cdot,\cdot) < 0$, plus the Inada conditions $\lim_{z\to 0}\mathcal{F}_z = \infty$ and $\lim_{z\to\infty}\mathcal{F}_z = 0$, where $z = K, N$.

The static profit maximization problem of the representative firm is

$$\Pi_t = \mathbf{Max}_{K_t, N_t} \left[Y_t - r_t K_t - w_t N_t \right], \tag{3.1}$$

where Π_t, r_t, and w_t denote firm profits, the real rental rate of capital, and the real wage (both in units of consumption). The first-order necessary conditions (FONCs) are

$$r_t = \mathcal{F}_K\left(K_t, A_t N_t\right) \text{ and } w_t = \mathcal{F}_N\left(K_t, A_t N_t\right) A_t. \tag{3.2}$$

The real rental rate of capital equals the marginal product of capital and the marginal product of labor is the real wage. These (relative) prices and the CRS technology yield zero equilibrium profits for the firm.

3.a.ii *The Household Sector*

Households take an address, $\jmath$, on the unit interval and possesses the period utility function $\mathcal{U}\left(c_t, \ell_t\right)$, where c_t and ℓ_t are consumption and leisure. The restrictions on period utility are $\mathcal{U}_c\left(\cdot, \cdot\right) > 0$, $\mathcal{U}_{cc}\left(\cdot, \cdot\right) < 0$, $\mathcal{U}_\ell\left(\cdot, \cdot\right) > 0$, $\mathcal{U}_{\ell\ell}\left(\cdot, \cdot\right) < 0$, along with the Inada conditions $\lim_{z \longrightarrow 0} U_z = \infty$ and $\lim_{z \longrightarrow \infty} U_z = 0$, where $z = c, \ell$. Households are endowed with one unit of time during each date t that is split between leisure and labor supply, h_t, $1 = \ell_t + h_t$. The $\jmath$th household faces the budget constraint

$$w_t(\jmath)h_t(\jmath) + r_t k_t(\jmath) + b_t(\jmath) + \Pi_t(\jmath) - T_t = c_t(\jmath) + x_t(\jmath) + q_t b_{t+1}(\jmath), \tag{3.3}$$

where x_t is investment, q_t is the price households pay to purchase unit discount bonds $b_{t+1}(\jmath)$, and T_t is a lump sum tax the government levies on all households. The law of motion of capital

$$k_{t+1}(\jmath) = x_t(\jmath) + (1 - \delta)k_t(\jmath), \quad \delta \in (0, 1), \tag{3.4}$$

sums investment and the capital that remains subsequent to its depreciation created by production. The household is able without cost to transform the single consumption into capital and reverse the process (*i.e.*, a putty-putty world).

The household is forward looking when maximizing its expected lifetime utility

$$\mathbf{E}_t \left\{ \sum_{t=0}^{\infty} \beta^t \mathcal{U}\left(c_t(\jmath),\, \ell_t(\jmath)\right) \right\}, \quad \beta \in (0,1),$$

subject to the budget constraint (3.3) and the law of motion of capital (3.4), given k_0. The mathematical expectations operator, $\mathbf{E}_t\{\cdot\}$, condition on date t information. The information set includes the history of the exogenous shocks and endogenous prices up to and including their date t realizations.

There are several approaches to solve this maximization problem. It is best to leave discussion of these important methods to Stokey and Lucas (1989), Malliaris and Brock (1998), Ljungqvist and Sargent (2012), and Kursell (2014). Some knowledge of these methods will be assumed in the remainder of these notes. In this example, the calculus of variation is applied to the dynamic Lagrange problem of the household

$$\begin{aligned} \mathscr{L}_t \;=\; \mathbf{E}_t \Bigg\{ \sum_{j=0}^{\infty} \beta^{t+j} \mathcal{U}\left(c_{t+j}(\jmath),\, 1 - h_{t+j}(\jmath)\right) \;+\; \sum_{j=0}^{\infty} \lambda_{t+j} \Big[w_{t+j} h_{t+j}(\jmath) \;+\; b_{t+j}(\jmath) \;+\; \Pi_{t+j}(\jmath) \\ -\; T_{t+j} \;-\; \left[c_{i,t+j} \;+\; k_{i,t+j+1} \;-\; \left(r_{t+j} \;+\; 1 \;-\; \delta\right) k_{i,t+j} \;+\; q_{t+j} b_{i,t+j+1} \right] \Big] \Bigg\}, \end{aligned}$$

where λ_t is the Lagrange multiplier or shadow price of one unit of the consumption good. The household maximizes the value of its lifetime utility program through the choice of uncertain streams of consumption, labor supply, capital, and bonds. The FONCs with respect to $c_t(j)$, $h_t(\jmath)$, $k_{t+1}(\jmath)$, and $b_{t+1}(\jmath)$ are

$$\beta^t \mathcal{U}_{c,\jmath,t} \;-\; \lambda_t \;=\; 0, \tag{3.5}$$

$$-\beta^t \mathcal{U}_{\ell,\jmath,t} \;+\; \lambda_t w_t \;=\; 0, \tag{3.6}$$

$$-\beta^t \mathcal{U}_{c,\jmath,t} \; + \; \mathbf{E}_t \left\{ \lambda_{t+1} \left[r_{t+1} \; + \; \left(1 - \delta \right) \right] \right\} \; = \; 0, \tag{3.7}$$

and

$$-q_t \beta^t \mathcal{U}_{c,\jmath,t} \; + \; \mathbf{E}_t \lambda_{t+1} \; = \; 0, \tag{3.8}$$

respectively, where for example $U_{c,\jmath,t} \; \equiv \; \mathcal{U}_c \left(c_t(\jmath), \, \ell_t(\jmath) \right)$.

The FONC have standard interpretations. Equation (3.5) states the Lagrange multiplier equals the marginal utility of consumption discounted back to date zero. The value of an additional unit of consumption is its value in utils. The marginal utility of leisure equals the value an extra unit of labor income yields to the household. The combination of these two FONC produces the intratemporal optimality condition

$$w_t \; = \; \frac{\mathcal{U}_{\ell,\jmath,t}}{\mathcal{U}_{c,\jmath,t}}. \tag{3.9}$$

Optimal consumption and labor supply choices leads the household to set its marginal rate of substitution between leisure and consumption equal to the real wage.

The intertemporal FONC (3.7) describes the trade-off the household faces when it postpones current consumption and holds additional capital. Current utility is lower by the drop in consumption, $\mathcal{U}_{c,t} \; = \; \lambda_t / \beta^t$, where the household address $\jmath$ is dropped from the subscript of the marginal utility of consumption. The shadow price of a unit of the consumption good and the discount factor is common across households. The discounted marginal utility of consumption equals the expected discounted value, in utils, of the date $t + 1$ rental income from capital, r_{t+1}, plus the fraction of the additional capital that remains after production, $1 \; - \; \delta$. Rewrite the intertemporal FONC (3.7) as the Euler equation

$$\mathcal{U}_{c,t} \; = \; \beta \mathbf{E}_t \left\{ \mathcal{U}_{c,t+1} \left[r_{t+1} \; + \; (1 - \delta) \right] \right\}. \tag{3.10}$$

A similar argument generates the Euler equation for bonds

$$q_t \mathcal{U}_{c,t} \;=\; \beta \mathbf{E}_t \mathcal{U}_{c,t+1}. \tag{3.11}$$

The cost to the household of buying a unit discount bond today, q_t, valued at $\mathcal{U}_{c,t}$ equals the discounted expected benefit of the bond in the form of the additional discounted utils a unit of consumption is anticipated to bring the household next period. This optimality condition defines the pricing kernel or stochastic discount factor, $\beta \mathbf{E}_t \left\{\mathcal{U}_{c,t+1}/\mathcal{U}_{c,t}\right\}$, which is the rate at which the household is willing to substitute consumption intertemporally.

The transversality condition

$$\lim_{j \longrightarrow \infty} \beta^j \mathbf{E}_t \Big\{ \mathcal{U}_{c,t+j} K_{t+j} \Big\} \;=\; 0,$$

is the sufficient condition for an equilibrium to exist for this RBC model. The household cannot be better off holding an extra unit of capital forever. Eventually, the household can only improve its welfare by consuming the capital. Establishing necessary conditions involves more analysis, which is left for Stokey and Lucas (1989), Ljungqvist and Sargent (2012), and Kursell (2014).

3.a.iii *The Government*

The government does not issue nominal liabilities (*i.e.*, money, treasury bills, and/or bonds). Period by period the government spends G_t units of the consumption good that is assumed to have no impact on the economy. Government spending is a stochastic process that is realized at the beginning of date t, which implies it is unknown by all participants in the economy at earlier dates. Since the government must obey its budget constraint, it follows that $T_t = G_t$. The government's primary budget is in balance period by period.

3.a.iv *Market Clearing, Equilibrium, and Optimality*

The three sectors of the economy are tied together by equilibrium conditions. These conditions define market clearing in the labor, capital, bond, and goods markets, which are

$$\text{Labor:}\ \ N_t \ = \ H_t,\ H_t \ \equiv \ \int_0^1 h_t(\jmath)di,\ \text{for } w_t \ > \ 0,\ \text{otherwise } w_t \ = \ 0,$$

$$\text{Capital:}\ \ K_t \ = \ k_t,\ k_t \ \equiv \ \int_0^1 k_t(\jmath)di,\ \text{for } r_t \ > \ 0,\ \text{otherwise } r_t \ = \ 0,$$

$$\text{Bond:}\ \ B_{t+1} \ = \ 0,\ B_{t+1} \ \equiv \ \int_0^1 b_{t+1}(\jmath)di,\ \text{for } q_t \ > \ 0,\ \text{otherwise } q_t \ = \ 0,$$

and

$$\text{Goods:}\ \ Y_t \ = \ C_t \ + \ X_t \ + \ G_t,\ \ C_t \ \equiv \ \int_0^1 c_t(\jmath)di \ \ \text{and} \ \ X_t \ \equiv \ \int_0^1 x_t(\jmath)di.$$

Labor, capital, and bond market clearing and the firm's and household optimality conditions implies optimality in general equilibrium

$$\mathcal{F}_N\left(K_t,\, A_tN_t\right)A_t \ = \ \frac{\mathcal{U}_\ell\left(C_t,\, 1-N_t\right)}{\mathcal{U}_c(C_t,\, 1-N_t)}, \tag{3.12}$$

$$1 \ = \ \beta\mathbf{E}_t\left\{\left[\frac{\mathcal{U}_c\left(C_{t+1},\, 1-N_{t+1}\right)}{\mathcal{U}_c\left(C_t,\, 1-N_t\right)}\right]\left[\mathcal{F}_K\left(K_{t+1},\, A_{t+1}N_{t+1}\right) \ + \ \Big(1 \ - \ \delta\Big)\right]\right\}, \tag{3.13}$$

and

$$q_t \ = \ \beta\mathbf{E}_t\left\{\frac{\mathcal{U}_c\left(C_{t+1},\, 1-N_{t+1}\right)}{\mathcal{U}_c\left(C_t,\, 1-N_t\right)}\right\}, \tag{3.14}$$

respectively. Only three optimality are needed because goods market clearing follows by applying the other three market clearing rules (a result of Walrus' law), equilibrium profits, and the government's budget constraint to the household's budget constraint (3.3).

Optimality in general equilibrium describes the behavior of the aggregate economy. In the labor market, the intratemporal optimality condition (3.12) sets the marginal product of aggregate labor input equal to the marginal rate of substitution between leisure and consumption in the household sector. The Euler equation (3.13) restricts the intertemporal equilibrium path of the aggregate economy. Optimality for the aggregate economy requires the value of one unit of the consumption-capital good today to equal the discounted expected value that good generates in production at date $t + 1$. The value is measured at the the marginal rate of substitution of C_t for C_{t+1}. The final aggregate optimality condition (3.14) prices this marginal rate of substitution in the bond market. It is the non-state contingent price of a one unit of the consumption good at date $t + 1$.

3.b *A Note on a Rational Expectations Equilibrium*

The RBC model needs an equilibrium definition to account for its characteristics. First, observe that households know nothing about the decision making process of the firm. All households observe are the relative prices $\{w_t, r_t, q_t\}_{t=0}^{\infty}$ called out by the Walrasian auctioneers in the perfectly competitive labor, capital, and bond markets. Given these prices and G_t, households supply h_t and k_t to the labor and goods market. The supplies of h_t and k_t are upward sloping functions of w_t and r_t. Likewise, the firm demands N_t and K_t when faced with w_t and r_t, given the realization of A_t. These demand schedules are downward sloping in the relative prices.

The important point is that all economic agents take relative prices and exogenous shocks as given when making optimal decisions in this RBC model with perfectly competitive markets. The more subtle point is that agents view these endogenous and exogenous objects as being drawn from a multivariate probability distribution. For the equilibrium of a RBC model to be well posed, the subjective beliefs economic agents possess about this multivariate probability distribution must match the true process. This is the fundamental idea of a REE.

Definition: A REE requires the subjective beliefs the representative household and the typical firm hold about the stochastic process

$$\mathcal{Z}_{t=0}^{\infty} = \{A_t,\ G_t,\ w_t,\ r_t,\ q_t\}_{t=0}^{\infty},$$

and the optimal choices $\{c_t,\ h_t,\ k_{t+1},\ B_{t+1}\}_{t=0}^{\infty}$ by the representative household and $\{K_t,\ N_t\}_{t=0}^{\infty}$ by the typical firm satisfy the labor, capital, bonds, and goods market clearing conditions. Since these choices by households and firms reflect their subjective beliefs over the stochastic process $\mathcal{Z}_{t=0}^{\infty}$, a REE require these subjective beliefs to match the true stochastic process that generates $\mathcal{Z}_{t=0}^{\infty}$.

The definition of a REE does not state that agents have complete information about all aspects of the economy. Agents need know only the information required to solve optimally their constrained maximization (or minimization) problems. For example, the household and firm know conditional moments of the multivariate stochastic process that produce A_t and G_t. Hence the definition of a REE does not state that agents predict with perfection $A_t,\ G_t,\ w_t,\ r_t$, and q_t period by period. Instead, agents forecast "perfectly only on average". These are predictions of conditional moments, which gives a REE the flavor of a probability statement. There is also no claim that within a REE the forecast errors of agents are unpredictable. Suppose an econometrician had access to the expectations of agents generated within a REE. If tests show these expectations yield persistent forecast errors, this is evidence the forecasting model invoked by the econometrician produces inefficient predictions. This is a test of forecast efficiency not RE. The RE of agents exist within the context of a DSGE model in which they operate; see Muth (1961) and Lucas (1987).

3.c *Specification of Model Primitives*

A calibration depends on the specification of the primitives of preferences and technology. Assume the period utility function of the typical household has constant relative risk aversion and is non-separable in consumption and leisure

$$\mathcal{U}\left(C_t,\ 1 - N_t\right) \ = \ \frac{\left[C_t^{\phi}\ \left(1 - N_t\right)^{1-\phi}\right]^{1-\alpha}}{1\ -\ \alpha}, \tag{3.15}$$

which restricts $\alpha \neq 1$. In the case of $\alpha = 1$, the period utility function is additively separable in consumption and leisure

$$\mathcal{U}(C_t,\ 1 - N_t) \ = \ \phi \ln[C_t] \ + \ (1 \ - \ \phi) \ln[1 - N_t]\,, \tag{3.16}$$

when $\alpha \ = \ 1$ and in either case $\phi \in (0,\ 1)$. The CRS production technology is Cobb-Douglas

$$Y_t \ = \ K_t^{\theta}\Big(A_t N_t\Big)^{(1-\theta)}, \quad \theta \ \in \ (0,\ 1). \tag{3.17}$$

Labor augmenting technology change could be either a linear deterministic trend in time

$$\ln A_t \ = \ \gamma t \ + \ \ln a_t, \quad 0 \ < \ \gamma, \tag{3.18}$$

or a stochastic random walk (with drift) trend process

$$\ln a_t \ = \ \ln a_{t-1} \ + \ \varepsilon_t, \quad \varepsilon_t \ \sim \ \mathcal{N}\left(0,\ \sigma_\varepsilon^2\right). \tag{3.19}$$

The deterministic trend, γ, represents the deterministic trend rate of the growth of the economy. Stochastic growth is generated by the innovation ε_t of the random walk (3.19). It produces a permanent or trend effect because its impact on $\ln A_t$ never dies out.

The last bit of the model to specify is the exogenous government spending shock G_t. No matter if a deterministic trend or a stochastic trend drives the economy, the transitory shock to government spending, g_t, equals G_t/Y_t. Variation in the transitory component of government spending is around the government spending-output ratio. The transitory component of government spending is assumed to be a first-order autoregression

$$\ln g_t \ = \ (1 - \rho_g)\ln g^* \ + \ \rho_g \ln g_{t-1} \ + \ \eta_t, \quad |\rho_g| \ < \ 1, \quad \eta_t \ \sim \ \mathcal{N}\left(0,\ \sigma_\eta^2\right), \tag{3.20}$$

where g^* is the steady state or (population) mean value of the government spending-output ratio. Innovations to technology and government spending are uncorrelated at all leads and lags, $\mathbf{E}_t\left\{\varepsilon_{t+j}\,\eta_{t+s}\right\} = 0$, for all $j,\ s$.

3.d *Stochastic Detrending and Calibration*

Calibration starts by endowing the RBC model with a process for labor augmenting technology. When A_t follows the random walk with drift process (3.19), the aggregate quantities are stochastic detrending as

$$\widehat{K}_{t+1} = \frac{K_{t+1}}{A_t} \quad \text{and} \quad \widehat{Z}_t = \frac{Z_t}{A_t}, \quad \text{where} \quad Z = Y, C, \text{and } X.$$

The RBC model has a balanced growth path because the stochastic trend A_t is common to the aggregates Y_t, C_t, X_t, G_t, and K_{t+1}. Using the one-sector stochastic growth model to evaluate RBC theory is conditional on the maintained assumption that TFP is a random walk with drift. This is not necessarily an innocuous assumption. Canova and Ferroni (2011) argue that measurement error in the data and potential model misspecification, which includes the choice of the process generating TFP, can produce biased estimates of the theoretical business cycle moments obtained from a DSGE model.

Nonetheless, applying stochastic trending to the production technology (3.17) and the aggregate resource constraint creates

$$(1 - g_t)\,\widehat{Y}_t = (1 - g_t)\exp\Big\{-\theta\big(\gamma + \varepsilon_t\big)\Big\}\widehat{K}_t^{\theta} N_t^{1-\theta} = \widehat{C}_t + \widehat{X}_t, \tag{3.21}$$

The law of motion of the capital stock is altered by stochastic detrending

$$\widehat{K}_{t+1} = \frac{(1 - \delta)}{\exp\{\gamma + \varepsilon_t\}}\widehat{K}_t + \widehat{X}_t. \tag{3.22}$$

When period utility is (3.16), the optimality conditions (3.12), (3.13) and (3.14) become

$$(1 - \theta)\exp\Big\{-\theta\,(\gamma + \varepsilon_t)\Big\}\widehat{K}_t^{\theta} N_t^{-\theta} = (1 - \theta)\frac{\widehat{Y}_t}{N_t} = \left(\frac{1 - \phi}{\phi}\right)\frac{\widehat{C}_t}{1 - N_t}, \tag{3.23}$$

$$1 \;=\; \beta\mathbf{E}_t\left\{\left(\frac{\hat{C}_{t+1}}{\hat{C}_t}\right)^{-1}\left[\theta\exp\Big\{-\theta\left(\gamma+\varepsilon_{t+1}\right)\Big\}\hat{K}_{t+1}^{\theta-1}N_{t+1}^{1-\theta} + \frac{1-\delta}{\exp\{\gamma+\varepsilon_{t+1}\}}\right]\right\}$$

$$= \;\beta\mathbf{E}_t\left\{\left(\frac{\hat{C}_{t+1}}{\hat{C}_t}\right)^{-1}\left[\theta\frac{\hat{Y}_{t+1}}{\hat{K}_{t+1}} + \frac{1-\delta}{\exp\left\{\gamma+\varepsilon_{t+1}\right\}}\right]\right\}, \tag{3.24}$$

and

$$q_t \;=\; \beta\mathbf{E}_t\left\{\exp\Big\{-\left(\gamma+\varepsilon_{t+1}\right)\Big\}\left(\frac{\hat{C}_{t+1}}{\hat{C}_t}\right)^{-1}\right\}, \tag{3.25}$$

respectively.

The next step imposes the steady state, $Z^* = \tilde{Z}_t$ and $N^* = N_t$, for all dates t, on the stochastically detrended equilibrium and optimality conditions (3.21)–(3.25). Some algebra yields

$$\frac{X^*}{Y^*} \;=\; 1 - \frac{C^*}{Y^*} - g^*, \tag{3.26}$$

$$\frac{X^*}{K^*} \;=\; \frac{\gamma^* - (1-\delta)}{\gamma^*}, \quad \gamma^* \;\equiv\; \exp\{\gamma\}, \tag{3.27}$$

$$\frac{C^*}{Y^*} \;=\; \frac{\phi(1-\theta)}{1-\phi}\left(\frac{1-N^*}{N^*}\right), \tag{3.28}$$

$$\frac{K^*}{Y^*} \;=\; \frac{\beta\gamma^*\theta}{\gamma^* - \beta(1-\delta)}, \tag{3.29}$$

and

$$q^* \;=\; \frac{\beta}{\gamma^*}, \tag{3.30}$$

respectively.

The steady state system is recursive for this RBC model. Assuming the parameters and N^* are known, the steady state ratios in the capital and goods markets, X^*/K^*, C^*/Y^*, and K^*/Y^* are constructed using equations (3.27), (3.28), and (3.29). The first and third of these steady state ratios are used to compute X^*/Y^*. As a result, g^* is calculated in equation (3.26). Once the steady state ratios are computed, rearranging the steady state production technology (3.17) gives the steady state capital stock

$$K^* = \left[\exp\{-\theta\gamma\}\frac{K^*}{Y^*}\right]^{1/(1-\theta)} N^*, \tag{3.31}$$

from which the steady state levels of output, consumption, investment and government spending can be constructed using the steady state ratios.

A problem is the RBC model parameters are unknown. A calibration of the RBC model splits its parameter between those that have direct sample counterparts and those related to the theoretical moments restricted by the steady state equations (3.26)–(3.31). The deterministic growth rate of the economy, γ, the standard deviation of technology growth, σ_ε, the AR(1) coefficient of g_t, its mean, and its standard deviation, σ_η, are directly observed in the data. A productivity accounting exercise is used to produce a time series of A_t. The national income accounts of the OECD economies contain the needed real GDP, capital stock, and labor input data. Likewise, a time series for government spending is available to compute the coefficients of the AR(1) of g_t. For Canada and the U.S., this exercise produces

Table 1. Several Calibrated Parameters of the RBC Model

	γ	σ_ε	g^*	ρ_g	σ_η
Canada	0.00242	0.01199	0.23262	0.97383	0.01268
U.S.	0.00362	0.00947	0.18088	0.99062	0.01056

on a quarterly sample from $1965Q1$ to $1997Q4$.

This leaves θ, δ, β, and ϕ to choose. The steady state or population mean share of capital in aggregate income or output is equivalent to θ. The national income accounts of industrialized economies usual put this share between 30 to 40 percent. The value of the other technology parameter, δ, is determined from the observation that the annual average depreciation rate of the private fixed capital is somewhere between 7.5 and 10 percent. A sensible quarterly depreciation rate is 0.02 in light of this observation. Often, the subjective rate of time preference is calibrated to the sample mean of a (nearly) riskless asset such as three-month treasury bills. Steady state in the bond market then sets $\beta = \gamma^* q^*$ from the optimality condition (3.30). Given q^* equals 0.99105 and 0.99254 for Canada and the U.S., β equals 0.99345 and 0.99494, respectively. It is also legitimate to simply select a value of β, say, 0.995, which implies q^* given γ.

All that remains is to calibrate ϕ. The steady state optimality condition (3.28), which calculates the steady state consumption-output ratio, is a nonlinear function of θ, ϕ, and N^*. Since the steady state is short one optimality (or equilibrium) condition, a choice must be made. A common choice is to calibrate N^* to sample observations. For example, the steady state employment rates in Canada and the U.S. are 0.42074 and 0.43035. Given this, ϕ can be calibrated by setting the steady state consumption-output ratio to its sample average to solve the steady state labor market optimality condition (3.28). This yields ϕ equal to 0.37163 (0.40756) for Canada (the U.S.). Or ϕ can be calibrated to microeconometric estimates of labor supply, as in Kimmel and Kniesner (1998).

There are competing views of what calibration is and how it should be done. Prescott (1986) presents a fully worked out philosophy of calibrating RBC models. King, Plosser, and Rebelo (2002) is the technical appendix to King, Plosser, and Rebelo (1988a, 1988b), which were influential papers in the early RBC literature. Gregory and Smith (1990, 1991, 1996) and Watson (1993) view calibration experiments as econometric exercises aimed at estimation and/or testing DSGE models. A good source to learn about different philosophies of and strategies for calibration is the debate between Kydland and Prescott (1996), Hansen and Heckman (1996), and Sims (1996). Kehoe (2007) argues the approach described by Sims is useful for judging whether DSGE models can replicate conditional dynamic moments of data. Canova (1994), DeJong, Ingram, and Whiteman (1996), and Geweke (2010) develop Bayesian calibration methods. Prior predictive analysis is enlarged upon by Canova, Faust and Gupta (2010) engage posterior predictive analysis, and DeJong, Ingram, and Whiteman and Geweke employ, what Geweke's labels, the minimal econometric interpretation of DSGE models. Kano and Nason (2014) adapt the latter methods to assess the fit of new Keynesian DSGE models to sample spectral densities of sample consumption and output growth.

3.e *Linearizing and Solving the RBC Model*

Linearization methods construct decision rules for the endogenous state variables, given the calibration is complete and steady state is calculated. There are more than a few linearization methods available in macroeconomics. Among these are methods to solve linear RE models developed by Zadrozny (1998), King and Watson (1998, 2002), Klein (2000), and Sims (2001). Their essentials are all the same. First, compute the eigenvalue and eigenvectors of the stable roots of the endogenous state variables. Then construct the forward looking forecasts of the exogenous state variables.

For the RBC model at hand, the lone endogenous state variable is $\widehat{K}_{t+1}$. These notes use a numerical solution method that conjectures its log linear approximate decision rule

$$\widetilde{K}_{t+1} = \mu_K \widetilde{K}_t + \mu_V \widetilde{V}_t, \tag{3.32}$$

where the goal is to compute the unknown coefficients, μ_K and μ_V, $\widetilde{K}_{t+1} = \ln \widehat{K}_{t+1} - \ln K^*$, and $\widetilde{V}_t = [\varepsilon_t \;\; \widetilde{g}_t]'$. The conjectured solution is written in terms of deviations from the steady state. Hence, the solution captures the transition path of stochastically detrended capital to its steady state in response to innovations to technology and government spending shocks.

Zadrozny (1998) provides a solution for the scalar μ_K and the one-by-two matrix μ_Z. Adapting this solution algorithm to the log linear approximate decision rule (3.32) employs the method of undetermined coefficients. It relies on taking first-order Taylor expansions around the (log of the) steady state of the optimality and equilibrium conditions (3.21)–(3.24).

The system of control equations is produced by linearizing the stochastically detrended aggregate resource constraint and the intratemporal labor market optimality condition (3.23). First, substitute for $\widehat{X}_t$ in (3.21) with (3.22) and construct its linear approximation

$$\theta\widetilde{K}_t + (1-\theta)\widetilde{N}_t - \theta\varepsilon_t - \left(\frac{g^*}{1-g^*}\right)\widetilde{g}_t = Y_C\widetilde{C}_t + Y_{1,K}\widetilde{K}_{t+1} + Y_K\widetilde{K}_t + Y_\varepsilon\varepsilon_t, \tag{3.33}$$

where $Y_C = C^*/Y^*,\ Y_{1K} = K^*/Y^*,\ Y_K = -(1-\delta)Y_{1K}/\gamma^*$, and $Y_\varepsilon = -Y_K$. A similar operation on the intratemporal labor market optimality condition (3.23) yields

$$\theta\widetilde{K}_t - \theta\widetilde{N}_t - \theta\varepsilon_t = \widetilde{C}_t + \left(\frac{N^*}{1-N^*}\right)\widetilde{N}_t. \tag{3.34}$$

The linearized equations (3.33) and (3.34) form the matrix system in the control variables $\widetilde{C}_t$ and $\widetilde{N}_t$. The system is

$$\begin{bmatrix} Y_C & -(1-\theta) \\ 1 & \theta + \dfrac{N^*}{1-N^*} \end{bmatrix}\begin{bmatrix} \widetilde{C}_t \\ \widetilde{N}_t \end{bmatrix} = \begin{bmatrix} -Y_{1,K} \\ 0 \end{bmatrix}\widetilde{K}_{t+1} + \begin{bmatrix} \theta - Y_K \\ \theta \end{bmatrix}\widetilde{K}_t$$

$$+ \begin{bmatrix} -(\theta + Y_\varepsilon) & -\dfrac{g^*}{1-g^*} \\ -\theta & 0 \end{bmatrix}\begin{bmatrix} \varepsilon_t \\ \widetilde{g}_t \end{bmatrix}.$$

Define $\widetilde{\mathcal{W}}_t = [\widetilde{C}_t\ \ \widetilde{N}_t]'$ to write the control system as

$$\widetilde{\mathcal{W}}_t = \mathcal{W}_{1,K}\widetilde{K}_{t+1} + \mathcal{W}_K\widetilde{K}_t + \mathcal{W}_V\widetilde{V}_t, \tag{3.35}$$

where $\widetilde{V}_{t+1} = \rho_V\widetilde{V}_t + e_{t+1}$, ρ_V is a 2×2 matrix full of zeros expect ρ_g resides in its (2, 2) position and $e_t = [\varepsilon_t \;\; \eta_t]'$.

The final optimality condition to be linearized is the stochastically detrended Euler equation of capital (3.24)

$$\widetilde{C}_t \; = \; \mathbf{E}_t \left\{ \widetilde{C}_{t+1} \; + \; K_{1,K}\widetilde{K}_{t+1} \; + \; K_{1,N}\widetilde{N}_{t+1} \; + \; K_{1,\varepsilon}\varepsilon_{t+1} \right\}, \tag{3.36}$$

where $K_{1,K} = (1-\theta)K_K$, $K_{1,N} = -K_{1,K}$, $K_{1,\varepsilon} = \theta K_K + \beta^{-1}(1-\delta)/\gamma^*$, and $K_K = \beta^{-1}\theta/Y_{1K}$. Combining the linearized control system (3.35) and linearized Euler equation (3.36) results in

$$K_{1,K}\widetilde{K}_{t+1} \; = \; \begin{bmatrix} -1 & -K_{1,N} \end{bmatrix} \mathbf{E}_t \widetilde{\mathcal{W}}_{t+1} \; + \; \begin{bmatrix} 1 & 0 \end{bmatrix} \widetilde{\mathcal{W}}_t \; + \; \begin{bmatrix} -K_{1,\varepsilon} & 0 \end{bmatrix} \mathbf{E}_t \widetilde{V}_{t+1},$$

which is a second-order stochastic difference equation with first-order moving average shocks as shown in

$$\begin{aligned} K_{1,\mathcal{W}}\mathcal{W}_{1,K}\mathbf{E}_t\widetilde{K}_{t+2} \; + \; \left[K_{1,\mathcal{W}}\mathcal{W}_K \; + \; K_{\mathcal{W}}\mathcal{W}_{1,K} \; - \; K_{1,K}\right]\widetilde{K}_{t+1} \; + \; K_{\mathcal{W}}\mathcal{W}_K\widetilde{K}_t & \\ = \; -\left[K_{1,\mathcal{W}}\mathcal{W}_V \; + \; K_{1,V}\right]\mathbf{E}_t\widetilde{V}_{t+1} \; - \; K_{\mathcal{W}}\mathcal{W}_V\widetilde{V}_t, & \end{aligned}$$

where $K_{1,\mathcal{W}} = \begin{bmatrix} -1 & -K_{1,N} \end{bmatrix}$, $K_{\mathcal{W}} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and $K_{1,V} = \begin{bmatrix} -K_{1,\varepsilon} & 0 \end{bmatrix}$. Next, substitute for $\widetilde{K}_{t+2}$ and $\widetilde{K}_{t+1}$ with the conjectured decision rule (3.32) to obtain

$$\left[\mathcal{K}_2\,\mu_K^2 \; + \; \mathcal{K}_1\,\mu_K \; + \; \mathcal{K}_0\right]\widetilde{K}_t \; = \; \left[\mathcal{V}_0 \; - \; \mathcal{V}_1\mu_V \; - \; \left(K_{1,\mathcal{W}}\mathcal{W}_K\right)\mu_V\rho_V\right]\widetilde{V}_t, \tag{3.37}$$

where $\mathcal{K}_2 = K_{1,\mathcal{W}}\mathcal{W}_{1,K}$, $\mathcal{K}_1 = K_{1,\mathcal{W}}\mathcal{W}_K + K_{\mathcal{W}}\mathcal{W}_{1,K} - K_{1,K}$, $\mathcal{K}_0 = K_{\mathcal{W}}\mathcal{W}_K$, $\mathcal{V}_0 = -K_{\mathcal{W}}\mathcal{W}_V - (K_{1,V} + K_{1,\mathcal{W}}\mathcal{W}_V)\rho_V$, $\mathcal{V}_1 = K_{1,\mathcal{W}}\mathcal{W}_{1,K}\mu_K + K_{1,\mathcal{W}}\mathcal{W}_K + K_{\mathcal{W}}\mathcal{W}_{1,K} - K_{1,K}$. The polynomial coefficients on the right and left hand sides of the linearized equilibrium condition (3.37) represent zero conditions. These zero conditions are univariate, $0 = \mathcal{K}_2\mu_K^2 + \mathcal{K}_1\mu_K + \mathcal{K}_0$ and bivariate $\mathbf{0}_{2\times 1} = \mathcal{V}_0 - \mathcal{V}_1\mu_V - (K_{1,\mathcal{W}}\mathcal{W}_K)\mu_V\rho_V$ for (3.37) to be in equilibrium for the linearized RBC model, the dimensions of the zero conditions of the endogenous and exogenous states have the same dimensions of the endogenous state vector and the exogenous state vector.

The polynomial operator of $\widetilde{K}_t$ in (3.37) is a second-order AR in the zero condition

$$\mathcal{K}(\mu_K) \;=\; \mathcal{K}_2\mu_K^2 \;+\; \mathcal{K}_1\mu_K \;+\; \mathcal{K}_0. \tag{3.38}$$

Since the endogenous state vector of the RBC model is a scalar, $\widetilde{K}_t$, the polynomial coefficients are also scalars. An implication is the zero of equation (3.38) can be written in companion form

$$\begin{bmatrix} \mathcal{K}_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \widetilde{K}_{t+2} \\ \widetilde{K}_{t+1} \end{bmatrix} = \begin{bmatrix} -\mathcal{K}_1 & -\mathcal{K}_0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{K}_{t+1} \\ \widetilde{K}_t \end{bmatrix}.$$

The solution of the zero condition of the state are the eigenvalues of

$$\mathcal{Q} \;=\; \begin{bmatrix} \mathcal{K}_2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -\mathcal{K}_1 & -\mathcal{K}_0 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues exist and are either distinct and real or complex conjugates because $\mathcal{Q}$ is non-singular; see theorem 2 of Zadrozny (1998). The textbook algorithm to solve for its eigenvalues and eigenvectors is the Jordan decomposition, but modern computational software employ more sophisticated decomposition procedures. These algorithms are more efficient in the case of a RBC model that has a multivariate vector endogenous state vector. If $\mathcal{Q}$ is not full rank (*i.e.*, singular), theorem 3 of Zadrozny (1998) gives necessary conditions for a unique solution to the second-order polynomial AR in the zero condition $\mathcal{K}(\mu_K)$. King and Watson (1998, 2002), Klein (2000), and Sims (2001) also provide solution methods to solve linearized DSGE models in the case of a singular leading matrix in the AR process of the endogenous states.

Zadrozny (1998) discusses another method to solve for the zero of the AR of $\widetilde{K}_t$. The method constructs the implied second-order polynomial of $\mathcal{K}(\mu_K)$ in equation (3.38). The polynomial can be written and factored into the second-order difference equation

$$\mathcal{K}(\mathbf{L}) = \Big(1 - \xi_1\mathbf{L}\Big)\Big(1 - \xi_2\mathbf{L}\Big) = \mathcal{K}_0\left[1 + \frac{\mathcal{K}_1}{\mathcal{K}_0}\mathbf{L} + \frac{\mathcal{K}_2}{\mathcal{K}_0}\mathbf{L}^2\right].$$

This implies $\mathcal{K}_1/\mathcal{K}_0 = -(\xi_1 + \xi_2)$ and $\mathcal{K}_2/\mathcal{K}_0 = \xi_1\xi_2$. The combination of the two equations is the implicit function $\mathcal{K}(\xi_1) = \xi_1 + (\mathcal{K}_2/\mathcal{K}_0)\,\xi_1^{-1} - \mathcal{K}_1/\mathcal{K}_0$. Its FONC yields the solution that minimizes $\mathcal{K}(\xi_1)$, $\xi_1 = \sqrt{\mathcal{K}_2/\mathcal{K}_0}$.

These results are summarized in figure 8, which is adapted from Sargent (1987, p.202). The parabola that $\mathcal{K}(\xi)$ represents achieves $\mathcal{K}_1/\mathcal{K}_0$ at ξ_1, is minimized at $\sqrt{\mathcal{K}_2/\mathcal{K}_0}$, and at ξ_2 the parabola reaches $\mathcal{K}_1/\mathcal{K}_0$ once more. The parabola intersects a line of zero slope and equals $\mathcal{K}_1/\mathcal{K}_0$ at ξ_1 and ξ_2. Hence, the point of minimization of $\mathcal{K}(\xi)$ produces strict inequality restrictions on ξ_1 and ξ_2

$$0 < \xi_1 < \sqrt{\frac{\mathcal{K}_2}{\mathcal{K}_0}} < \sqrt{\beta^{-1}} \quad \text{and} \quad \sqrt{\frac{\mathcal{K}_2}{\mathcal{K}_0}} < \xi_2.$$

The polynomial $\mathcal{K}(\mathbf{L})$ has two roots (not necessarily distinct). The modulus of the root ξ_1 is in the open unit interval making it stable and backward-looking. The root ξ_2 is explosive and forward-looking because its modulus is greater than one and finite. Hence, the solution of $\mathcal{K}(\mathbf{L})$ is

$$(1 - \xi_1\mathbf{L})\left(1 - \frac{\mathcal{K}_2}{\mathcal{K}_0}\xi_2\mathbf{L}\right) = -\xi_2^{-1}\frac{\mathcal{K}_2}{\mathcal{K}_0}(1 - \xi_1\mathbf{L})\left(1 - \frac{\mathcal{K}_0}{\mathcal{K}_2}\xi_2^{-1}\mathbf{L}^{-1}\right)\mathbf{L}.$$

It is useful to understand the stable root of $\mathcal{K}(\mathbf{L})$, ξ_1, is the stable eigenvalue of $\mathcal{Q}$. The implication is $\mu_K = \xi_1$, which is the stable root of the decision rule (3.32).

The solution of the zeros of the exogenous state vector $\widetilde{V}_t$, the disturbances of the RBC model, requires some results from matrix algebra. Pass the vec operator through the zeros $\mathbf{0}_{2\times1} = \mathcal{V}_0 - \mathcal{V}_1\mu_V - (K_{1,\mathcal{W}}\mathcal{W}_K)\mu_V\rho_V$

$$\mathsf{vec}(\mathbf{0}_V) = \mathsf{vec}(\mathcal{V}_0) - \left[\mathbf{I}_2' \bigotimes \mathcal{V}_1\right]\mathsf{vec}(\mu_V) - \left[\rho_V' \bigotimes (K_{1,\mathcal{W}}\mathcal{W}_K)\right]\mathsf{vec}(\mu_V),$$

where the vec operators stacks the columns of a matrix on top of each one another moving from left to right, $\mathsf{vec}(D_1D_2D_3) = [D_3' \otimes D_1]\,\mathsf{vec}(D_2)$, where D_1, D_2, and D_3 are conformable and $\otimes$ is the Kronecker product. The two elements of μ_V are found by

$$\mathsf{vec}(\mu_V) = \left(\left[\mathbf{I}_2' \bigotimes \mathcal{V}_1\right] + \left[\rho_V' \bigotimes (K_{1,\mathcal{W}}\mathcal{W}_K)\right]\right)^{-1}\mathsf{vec}(\mathcal{V}_0),$$

where rearranging the left hand side of the equality produces the 2×2 impulse matrix of the conjectured linear approximate decision rule of (3.32).

The next step computes the other endogenous variables of the economy. First, define $\tilde{S}_{t+1} = [\tilde{K}_{t+1} \;\; \varepsilon_{t+1} \;\; \tilde{g}_{t+1}]'$ and $u_{t+1} = [0 \;\; \varepsilon_{t+1} \;\; \eta_{t+1}]'$. The linear approximate decision rule of (3.32) becomes the state space system

$$\tilde{S}_{t+1} = \begin{bmatrix} \mu_K & \mu_V \\ & \\ \mathbf{0}_{2\times1} & \rho_V \end{bmatrix}\tilde{S}_t + u_{t+1} = \mu_S\tilde{S}_t + u_{t+1}. \tag{3.39}$$

The control system (3.35) and the state space solution (3.39) yield

$$\tilde{\mathcal{W}}_t = \left[\left(\mathcal{W}_{1,K}\mu_K + \mathcal{W}_K\right) \quad \left(\mathcal{W}_{1,K}\mu_V + \mathcal{W}_V\right)\right]\tilde{S}_t = \Lambda_{\mathcal{W},S}\tilde{S}_t. \tag{3.40}$$

The remaining endogenous variables, which are investment and output, can added to the equilibrium solution of the control system (3.40) by linearizing the law of motion of capital (3.22) and the production technology or aggregate resources constraint (3.21).

The theoretical moments of $\widetilde{\mathcal{W}}_t$ of the linearized approximation of the RBC model are calculated with (3.39) and (3.40). The relative volatility and co-movement of the transitory components of the control system are found in the cross-covariance function of $\widetilde{\mathcal{W}}_t$

$$\mathbf{E}\left\{\widetilde{\mathcal{W}}_t\widetilde{\mathcal{W}}_{t-j}\right\} = \Lambda_{\mathcal{W},S}\mathbf{E}\left\{\widetilde{S}_t\widetilde{S}_{t-j}\right\}\Lambda'_{\mathcal{W},S} = \Lambda_{\mathcal{W},S}\mu_S\Sigma_S\Lambda'_{\mathcal{W},S},$$

where $\mathbf{E}\left\{\widetilde{S}_t\widetilde{S}_{t-j}\right\} = \mu_S^j\mathbf{E}\left\{\widetilde{S}_{t-j}\widetilde{S}_{t-j}\right\} \equiv \mu_S^j\Sigma_S$ by repeated substitution of the lags of (3.39) and the solution of the covariance matrix of the state space equation $\Sigma_S = \mu_S\Sigma_S\mu_S' + \Sigma_u$ is $\text{vec}\left(\Sigma_S\right)$ $= \left[\mathbf{I}_4 - \mu_S \otimes \mu_S\right]^{-1}\text{vec}\left(\Sigma_u\right)$. The theoretical responses of $\widetilde{S}_{t+1}$ and $\widetilde{\mathcal{W}}_t$ to the transitory technology and government spending shocks, ε_{t+1} and $\widetilde{g}_{t+1}$, are the partial derivatives

$$\frac{\partial\widetilde{S}_{t+j}}{\partial u_{t+1}} = \mu_S^{j-1} \quad \text{and} \quad \frac{\partial\widetilde{\mathcal{W}}_{t+j}}{\partial u_{t+1}} = \Lambda_{\mathcal{W},S}\mu_{\mathcal{Z}}^{j-1}, \tag{3.41}$$

because $\widetilde{S}_{t+j} = \sum_{i=0}^{\infty}\mu_S^i u_{t+j-i}$. The partial derivatives of equations (3.41) are theoretical impulse response functions (IRFs) of the transitory dynamics of $\widetilde{\mathcal{K}}_{t+1}$, $\widetilde{C}_t$, and $\widetilde{N}_t$.

Similar analysis produces the bivariate transitory dynamics of $\widetilde{Y}_t$ and $\widetilde{C}_t$. Begin by writing the linearized budget constraint (3.33) as

$$\widetilde{Y}_t - Y_C\widetilde{C}_t = Y_{1,K}\widetilde{K}_{t+1} + Y_K\widetilde{K}_t + Y_\varepsilon\varepsilon_t + \left(\frac{g^*}{1-g^*}\right)\widetilde{g}_t,$$

where the linearized stochastically detrended output technology is $\widetilde{Y}_t = \theta\widetilde{K}_t + (1-\theta)\widetilde{N}_t - \theta\varepsilon_t$. It along with rewriting the linearized intratemporal labor market optimality condition (3.34) as

$$\widetilde{N}_t = \left[\frac{1-N^*}{N^* + \theta(1-N^*)}\right]\left(\theta\widetilde{K}_t - \widetilde{C}_t - \theta\varepsilon_t\right)$$

yields a second control equation in $\widetilde{Y}_t$ and $\widetilde{C}_t$

$$\widetilde{Y}_t + \left[\frac{(1-\theta)(1-N^*)}{N^* + \theta(1-N^*)}\right]\widetilde{C}_t = \left[\frac{\theta}{N^* + \theta(1-N^*)}\right]\left(\widetilde{K}_t - \varepsilon_t\right).$$

Define $N_N \equiv [N^* + \theta(1 - N^*)]^{-1}$ to construct a linearized control system in $\widetilde{\mathcal{Y}}_t = \left[\widetilde{Y}_t \;\; \widetilde{C}_t\right]'$

$$\begin{bmatrix} 1 & -Y_C \\ \\ 1 & \theta_N \end{bmatrix} \widetilde{\mathcal{Y}}_t = \begin{bmatrix} Y_{1,K} \\ \\ 0 \end{bmatrix} \widetilde{K}_{t+1} + \begin{bmatrix} Y_K \\ \\ \theta N_N \end{bmatrix} \widetilde{K}_t + \begin{bmatrix} Y_\varepsilon & \dfrac{g^*}{1 - g^*} \\ \\ -\theta N_N & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ \\ \widetilde{g}_t \end{bmatrix}.$$

where $\theta_N \equiv (1 - \theta)(1 - N^*)N_N$ or in matrix form

$$\widetilde{\mathcal{Y}}_t = \mathcal{Y}_{1,K}\widetilde{K}_{t+1} + \mathcal{Y}_K\widetilde{K}_t + \mathcal{Y}_V\widetilde{V}_t. \quad (3.42)$$

The bivariate dynamics of $\widetilde{\mathcal{Y}}_t$ are studied by replacing the control system (3.40) with $\widetilde{\mathcal{Y}}_t = \Lambda_{\mathcal{Y},S}S_t$. The analysis is performed by substituting $\widetilde{\mathcal{Y}}_t$ and $\Lambda_{\mathcal{Y},S}$ into the theoretical cross-covariance function formula and IRFs (3.41) for $\widetilde{\mathcal{W}}_t$ and $\Lambda_{\mathcal{W},S}$.

The theoretical IRFs (3.41) are not equivalent to IRFs produced by estimating a univariate ARMA or a structural VAR on synthetic samples of $\Delta \ln Y_t$ or/and $\ln N_t$ created by simulating the linearized solution (3.39) and (3.40) of the RBC model. Cogley and Nason (1993) is an example showing this for the dynamics of $\Delta \ln Y_t$ using a similar RBC model.

A structural VAR can be estimated on artificial samples by simulating the linearized solution RBC model by restoring the stochastic trend to $\widetilde{Y}_t$ and $\widetilde{C}_t$. Remember $\widetilde{Z}_t = \ln \widehat{Z}_t - \ln Z^*$ and $\widehat{Z}_t = Z_t \big/ A_t$, where $Z = Y,\ C$, and X. Using these facts leads to

$$\mathcal{Y}_t = \mathcal{Y}^* + \Lambda_{\mathcal{Y},S}\widetilde{S}_t + \Lambda_{\mathcal{Y},A}\tau_t \quad (3.43)$$

where $\mathcal{Y}_t = [\ln Y_t \;\; \ln C_t]'$, $\mathcal{Y}^* = [\ln Y^* \;\; \ln C^*]'$, $\Lambda_{\mathcal{Y},A} = [1 \;\; 1]'$, and $\tau_t = \ln A_t$. The log level of the TFP shock is included in the system of state equations (3.39) by altering it to

$$\widetilde{\mathcal{T}}_{t+1} = \begin{bmatrix} \mu_K & 0 & \mu_V(1,2) \\ 0 & 1 & 0 \\ 0 & 0 & \rho_g \end{bmatrix} \widetilde{\mathcal{T}}_t + \begin{bmatrix} \mu_V(1,1) & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} e_{t+1} = \mu_{\mathcal{T}}\widetilde{\mathcal{T}}_t + \mu_e e_{t+1}, \quad (3.44)$$

where $\widetilde{\mathcal{T}}_{t+1} = [\widetilde{K}_{t+1} \quad \tau_{t+1} \quad \widetilde{g}_{t+1}]'$ and, for example, $\mu_V(1, 1)$ is the first element of the row vector μ_V. Simulating the state system (3.44) generates synthetic samples of $\widetilde{\mathcal{T}}_{t+1}$. Feeding these data into the control system (3.43) creates artificial samples of the levels of output and consumption in $\mathcal{Y}_t$. Structural VARs estimated on artificial samples of $\mathcal{Y}_t$ produce theoretical IRFs of the linearized RBC model that can be compared to sample IRFs. Sims (1996) and Kehoe (2007) argue these comparisons are useful for evaluating the dynamic fit of DSGE models to actual data. Tools to choose consistent and efficient IRFs are developed by Hall et al (2012). Cogley and Nason (1995b), Nason and Rogers (2006), and Kano and Nason (2014), among others, implement this approach to evaluating DSGE models.

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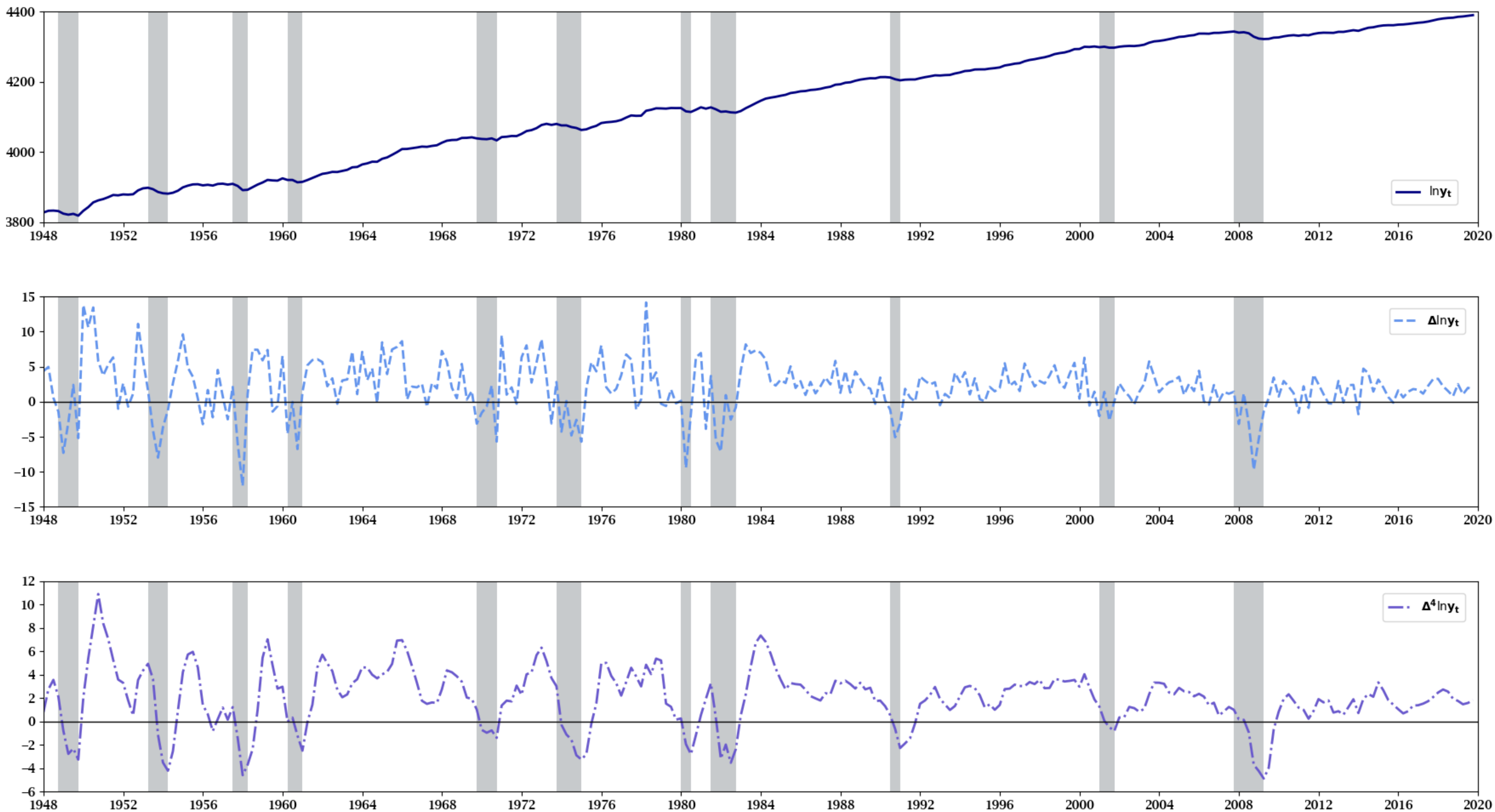
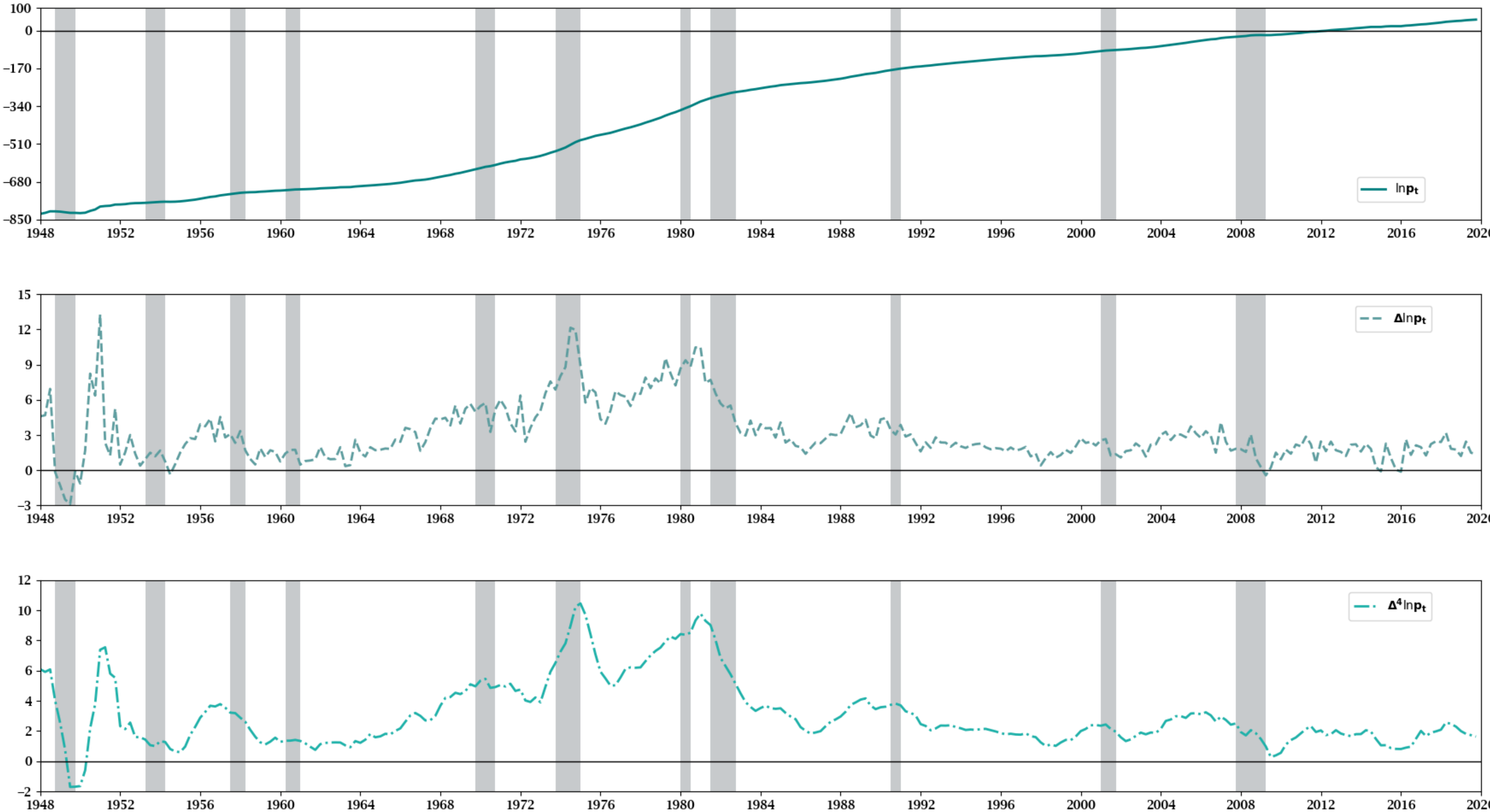
FIGURE 1: THE LOG LEVEL AND GROWTH RATES OF PER CAPITA REAL GDP, 1948Q1–2019Q4



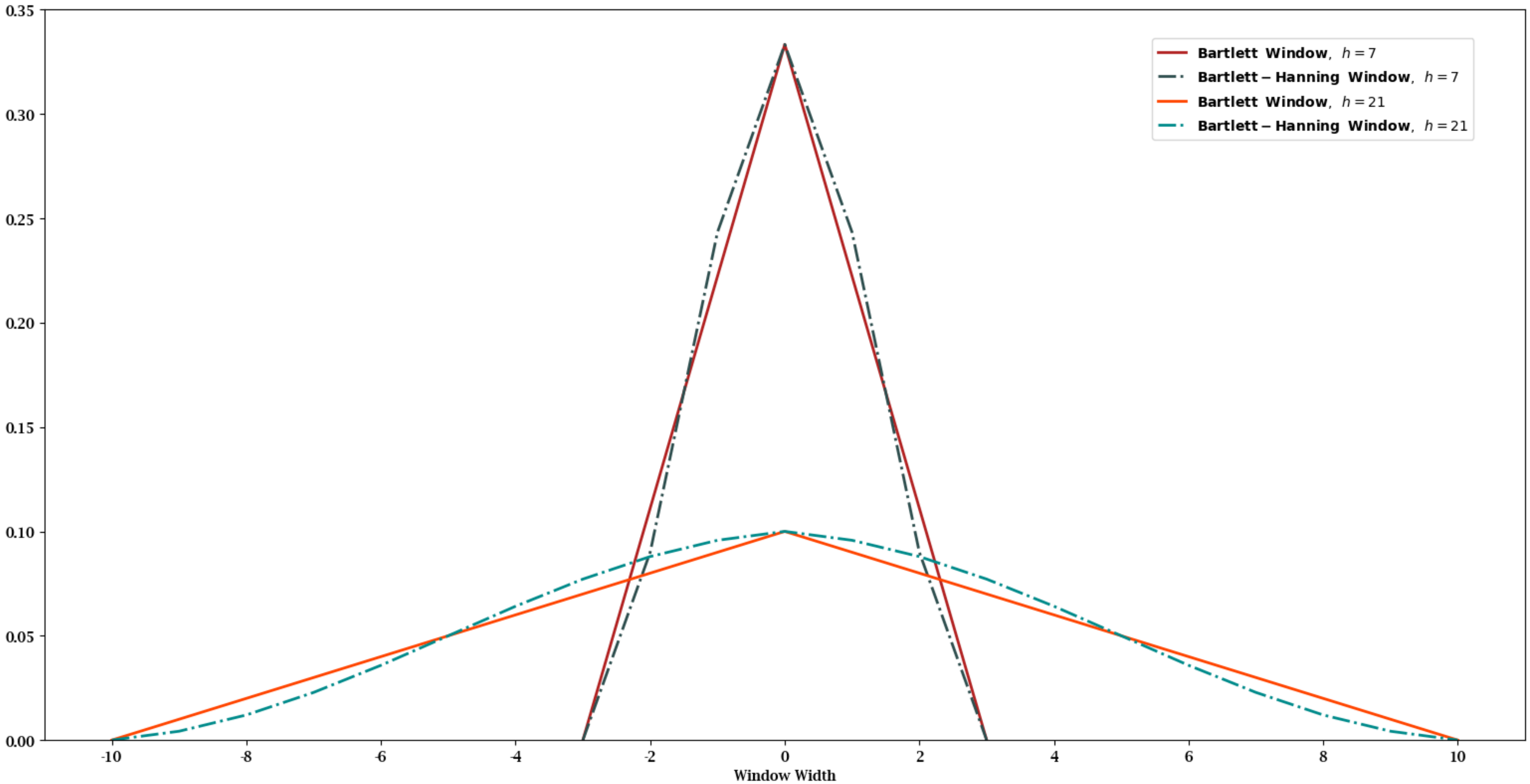
Notes: The top panel shows the log level of per capita real GDP. The middle panel display its quarter over quarter growth rate while the bottom panel depicts year over year growth rates. Vertical gray bands denote NBER dated recessions.

Figure 2: The Log Level and Growth Rates of the Chained Weighted GDP Deflator, 1948Q1–2019Q4



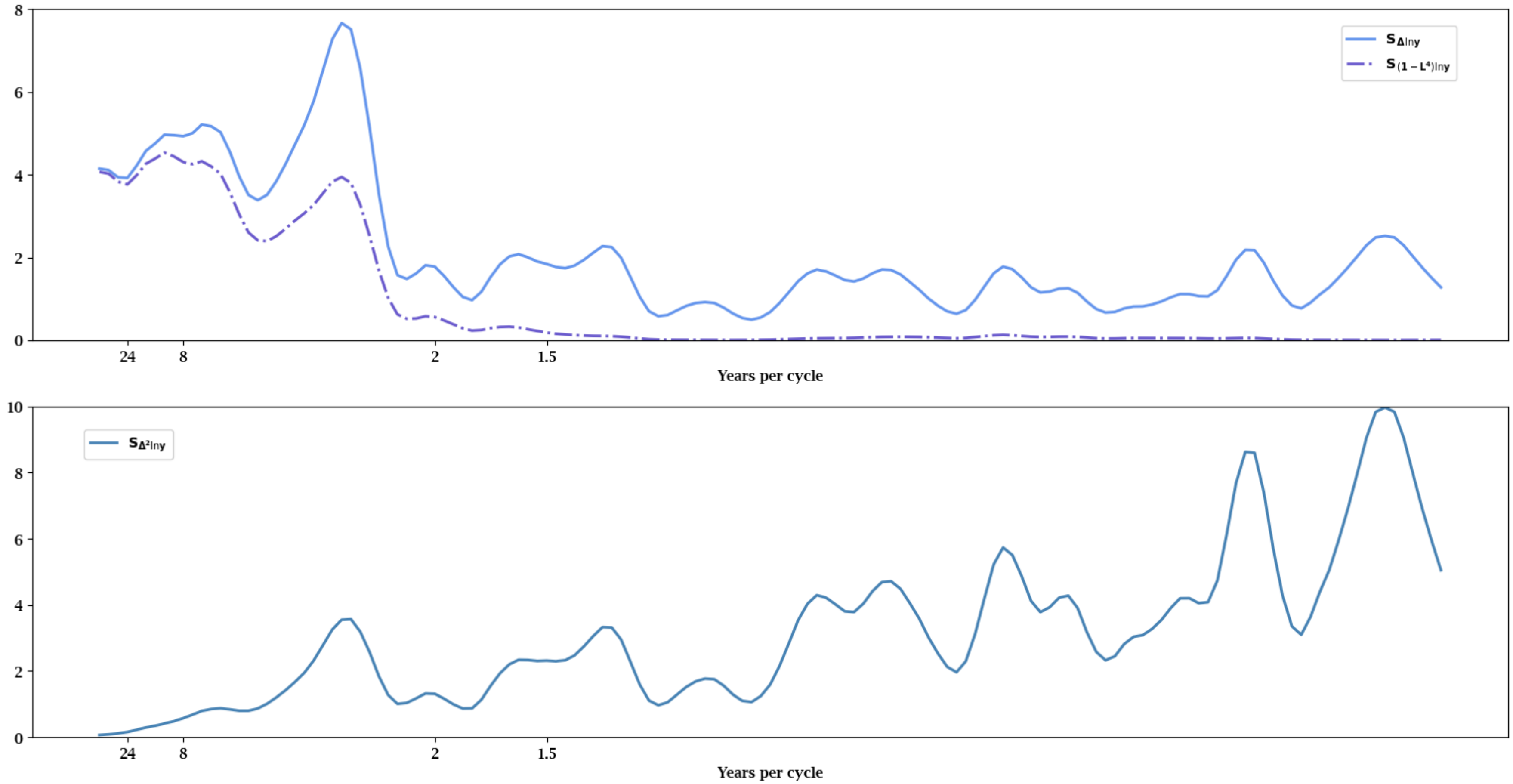
Notes: The top panel shows the log level of the chained weighted GDP deflator. The middle panel display its quarter over quarter growth rate as a measure of inflation. The bottom panel depicts inflation using year over year growth rates of the GDP deflator. Vertical gray bands denote NBER dated recessions.

Figure 3: Bartlett and Bartlett Hanning Windows at Widths $h = 7$ and 21



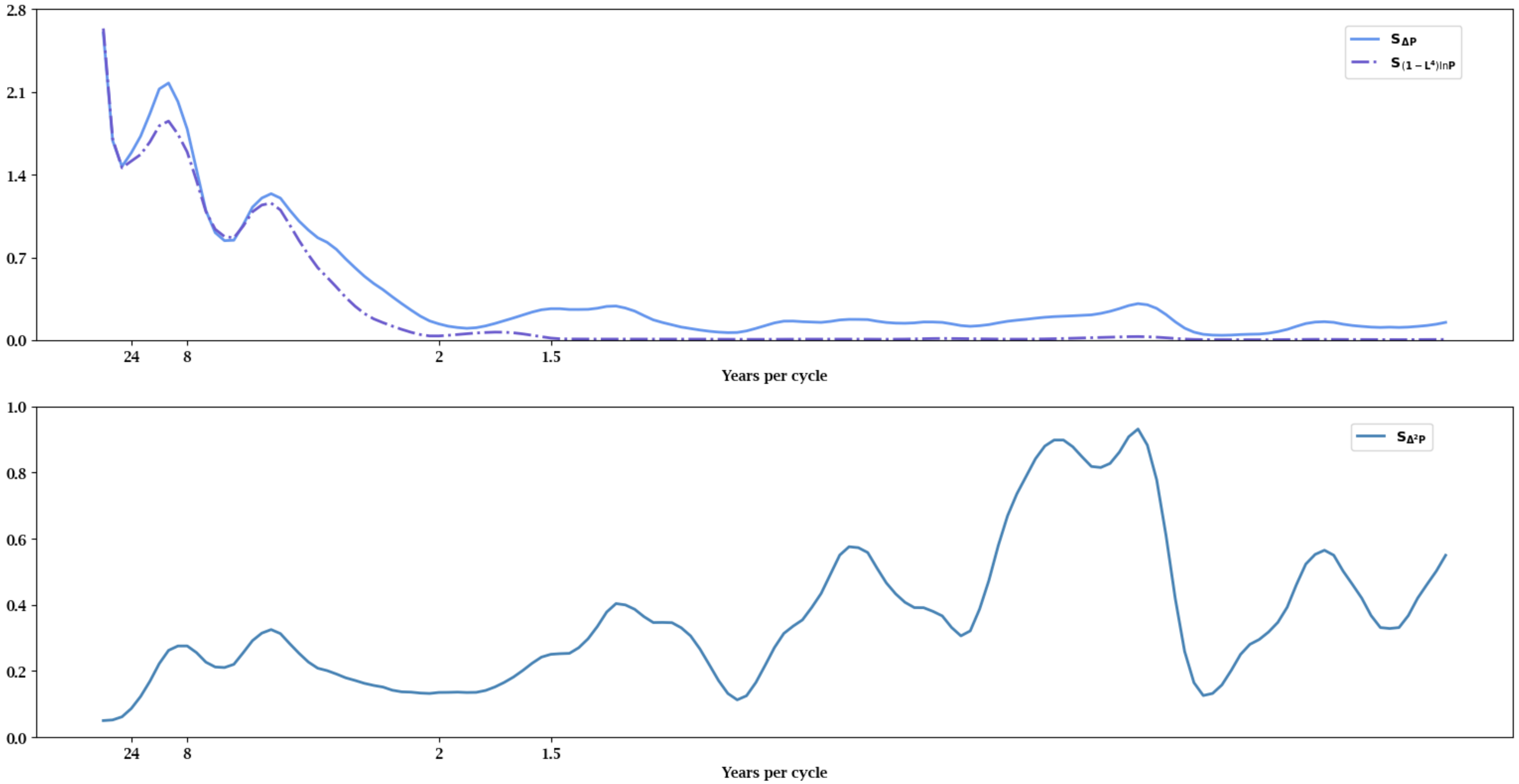
Notes: The Bartlett and Bartlett-Hanning windows are constructed using the Julia package `DSP.jl`.

FIGURE 4: SPECTRAL DENSITIES OF GROWTH RATES OF PER CAPITA REAL GDP, 1948Q1–2019Q4



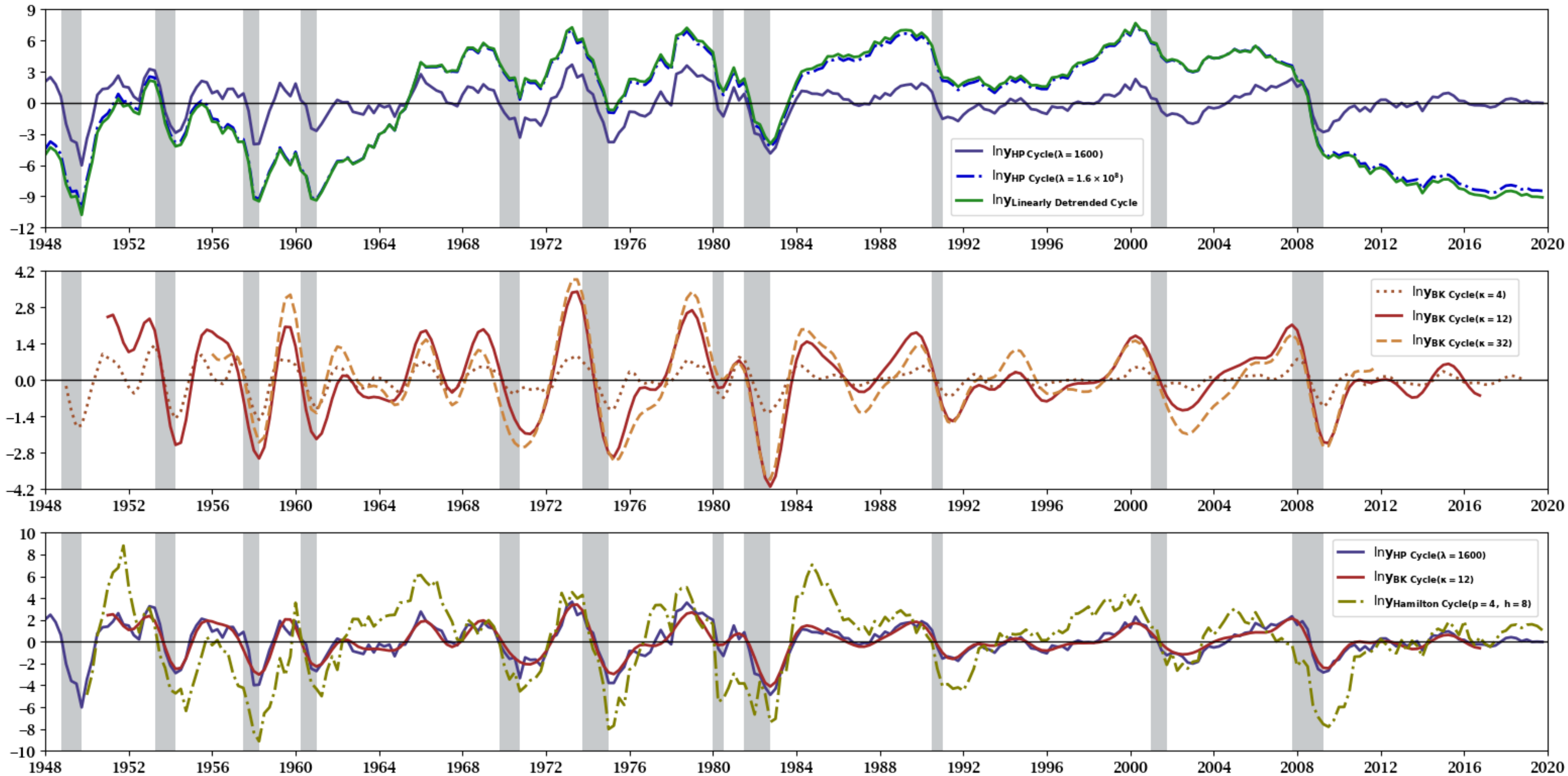
Notes: The top panel shows the spectral densities of per capita real GDP growth quarter over quarter and year over year. The bottom panel display the spectral density of the second difference of the log of per capita real GDP. The Bartlett-Hanning window is used to construct the smoothed spectral density with a window width of 11.

Figure 5: Spectral Densities of Chained Weighted GDP Deflator Inflation, 1948Q1–2019Q4



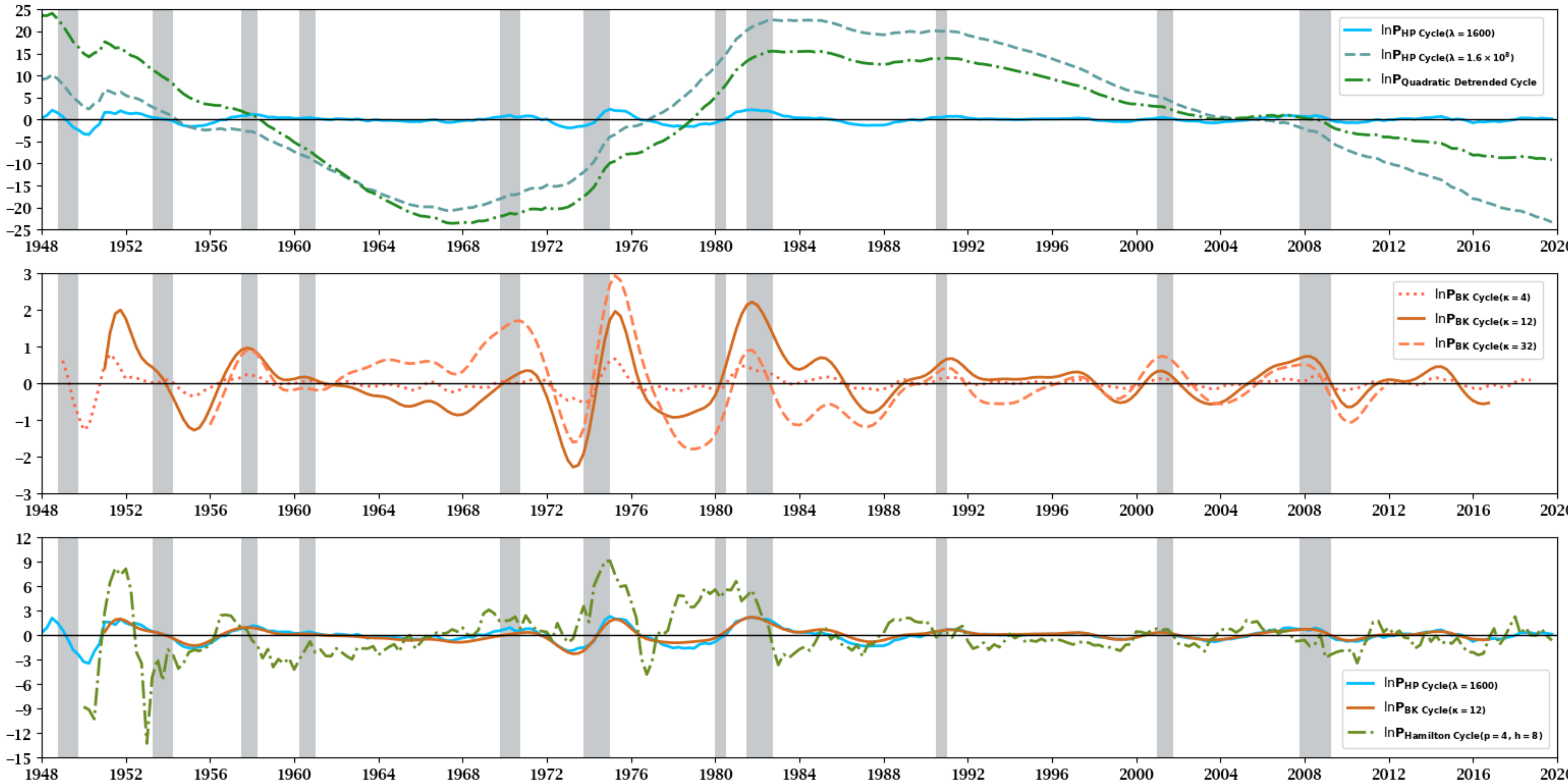
Notes: The top panel shows the spectral densities of inflation computed quarter over quarter and year over year using the log level of the chained weighted GDP deflator. The bottom panel display the spectral density of the first difference of chained weighted GDP deflator inflation. The bottom panel depicts inflation using year over year growth rates of the GDP deflator. The Bartlett-Hanning window is used to construct the smoothed spectral density with a window width of 11.

Figure 5: Hodrick-Prescott, Baxter-King, and Hamilton Output Gaps, 1948Q1–2019Q4



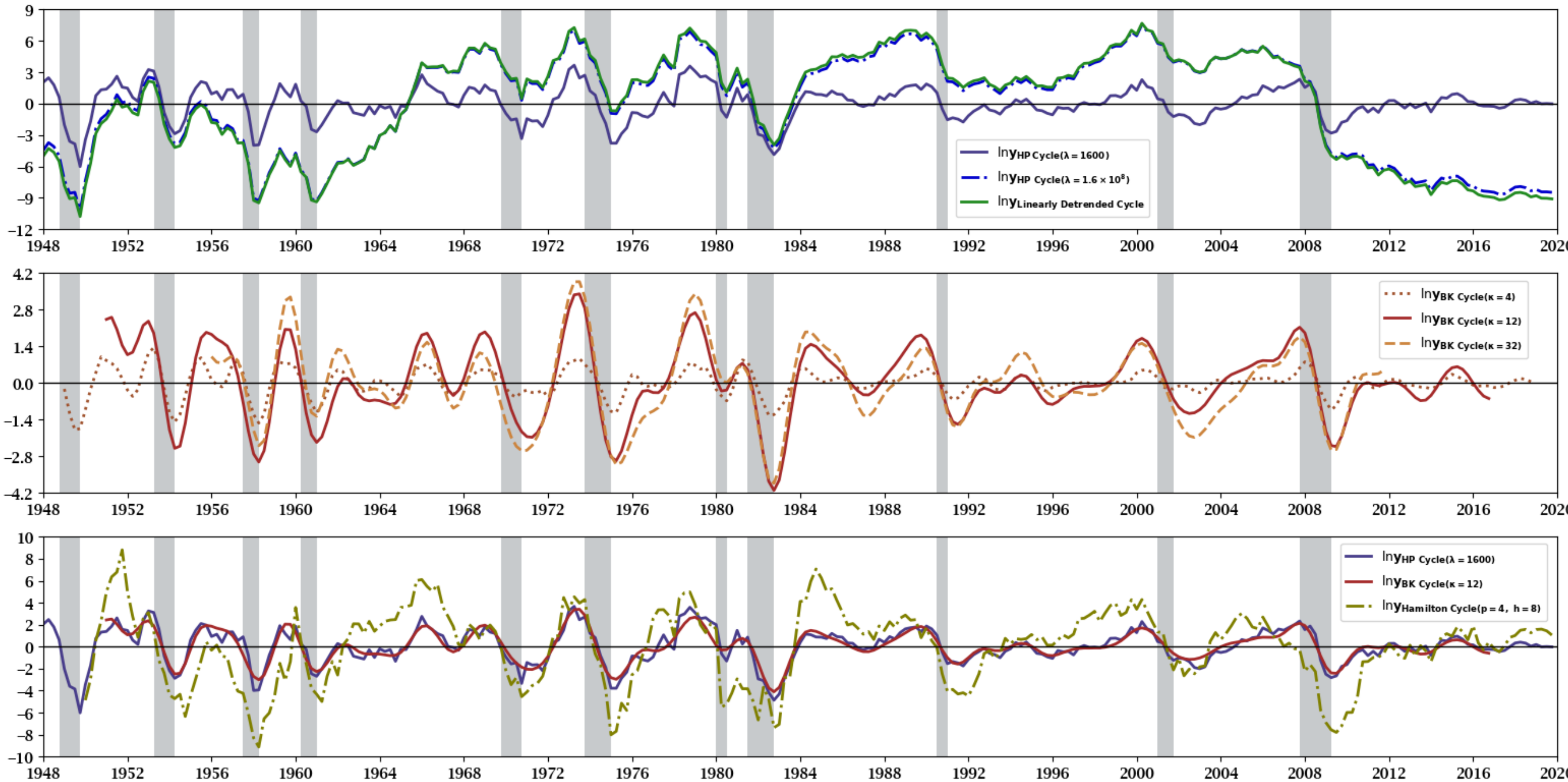
Notes: The top panel displays the Hodrick-Prescott output gap for different values of the smoothing parameter λ along with linearly detrended per capita real GDP growth quarter. Baxter-King output gap for different values of the lead-lag window width, κ appear in the middle panel. The bottom panel contains the Hodrick-Prescott output gap for $\lambda = 1600$, the Baxter-King output cycle for $\kappa = 12$, and the Hamilton output gap for and $h = 8$ and $p = 4$.

Figure 6: Hodrick-Prescott, Baxter-King, and Hamilton Price Level Gaps, 1948Q1–2019Q4



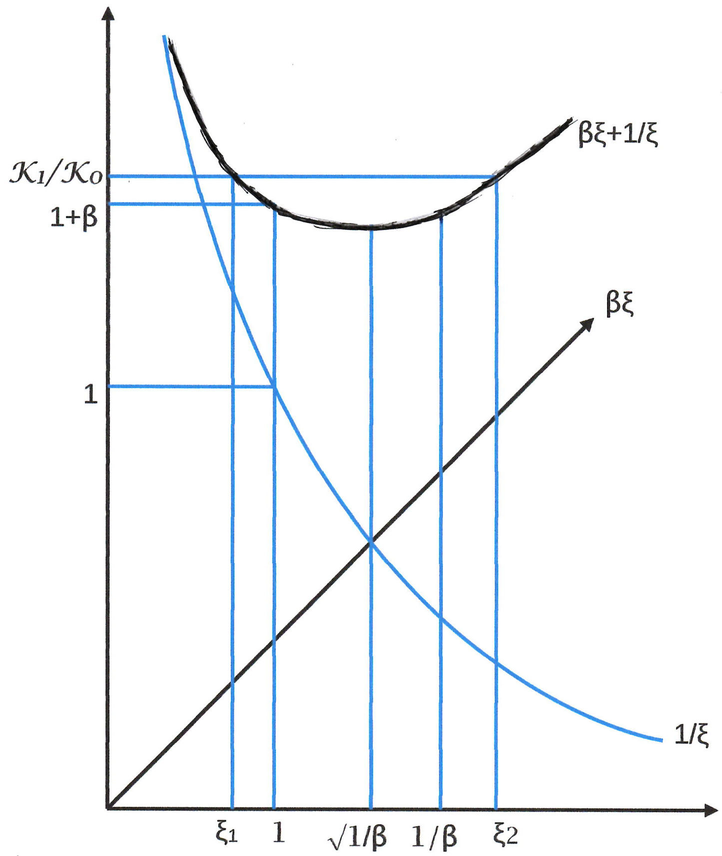
Notes: The top panel displays the Hodrick-Prescott price level gap for different values of the smoothing parameter λ along with the quadratic detrended chain weighted GDP price deflator. Baxter-King price gaps for different values of the lead-lag window width, κ appear in the middle panel. The bottom panel contains the Hodrick-Prescott price level gap for $\lambda = 1600$, the Baxter-King price level gap for $\kappa = 12$, and the Hamilton price level gap for and $h = 8$ and $p = 4$.

Figure 7: Hodrick-Prescott, Baxter-King, and Hamilton Inflation Gaps, 1948Q1–2019Q4



Notes: The top panel displays the Hodrick-Prescott inflation gap for different values of the smoothing parameter λ along with linearly detrended detrended chain weighted GDP price inflation. Baxter-King inflation gaps for different values of the lead-lag window width, κ appear in the middle panel. The bottom panel contains the Hodrick-Prescott inflation gap for $\lambda = 1600$, the Baxter-King output gap for $\kappa = 12$, and the Hamilton inflation gap for and $h = 8$ and $p = 4$.

Figure 8: Solution to the Second-Order Polynomial $\mathcal{K}(\mathbf{L})$



Notes: The modulus of the stable root, ξ_1, is on the open unit interval $(0, 1)$ while for unstable root, ξ_2, it is in $(1, \infty)$. The household discount factor is $\beta \in (0, 1)$. The coefficient on the first-order term in $\mathcal{K}(\mathbf{L})$ is $\mathcal{K}_1/\mathcal{K}_0 = -(\xi_1 + \xi_2)$.