Lecture 2: Financial Frictions and the Real Economy

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Liquidity Preference, Asset Prices, & Bank Runs

Allen and Gale (Chapters 2.3 & 3)
Gertler and Kiyotaki (2013, NBER wp–19129)
Kiyotaki and Moore (IER, 2005)
Jaccard (2013, or ECB WP1525, 2012)

Financial Fragility, Liquidity, and Collateral

Allen and Gale (Chapter 5)
Geanakoplos (NBER Macro Annual, 2009)
Guerrieri and Lorenzoni (NBER WP 17583, 2011)

Collateral, Liquidity, and Asset Prices in DSGE

Holmström & Tirole (2011, ch. 4): Liquidity Asset Pricing
Jermann and Quadrini (JME, 2007)
Bigio (Manuscript, Columbia B-School, U. of Columbia, 2012)
Shi (U. of Toronto, wp459, 2012)

Risk, Leverage, and Asset Prices in a DSGE Model

Brunnermeier and Sannikov (AER, 2014)

Introduction

- There are many definitions of liquidity.
  1. A liquid asset is one that can be sold quickly at its "fundamental value" $\Longrightarrow$ assets not subject to fire sales.
  2. Economic agents demand liquidity (*i.e.*, liquid assets) if the timing of their income is uncertain $\Longrightarrow$ liquid assets aid in smoothing consumption.

- Difficult to promise future income in trade for consumption today.
  1. Moral hazard and/or adverse selection makes promises of future payouts impossible $\Longrightarrow$ the principal-agent problem.
  2. Or can their beliefs about the future expected change? $\Longrightarrow$ Not a change in risk aversion, but changes in expectations predict changes in the demand for liquid assets.

- The demand for liquid assets suggests that prices of these securities will be higher (or lower yields) than less liquid assets $\Longrightarrow$ the liquidity premium, which is also known as the "risk-free rate puzzle."

A Model of Liquidity Shocks

- A three period economy, $t = 0, 1, 2$, consisting of households and a 1-period risk-free liquid asset, and a risky 2-period asset.
- Households, who are risk averse, consume at date 1 or date 2, but
  1. their endowment is received at date $t = 0$
  2. as one unit of the consumption good.
  3. With probability λ, households consume at date $t = 1$ and with probability $1 - \lambda$ they consume at date $t = 2$ $\Longrightarrow \mathbf{E}\Big\{\mathcal{V}(C_1, C_2)\Big\} = \lambda\mathcal{U}(C_1) + (1-\lambda)\mathcal{U}(C_2)$.
  4. At date 0, households know λ, but not their type.
- The risk-free asset is a technology that returns a unit of the consumption good tomorrow for every unit invested today.
- The long-dated asset is a technology that for every unit of the consumption good invested at date t returns $1 + R$, $R > 0$, at date $t+2$.

Liquidity Preference

- Households view the realization of the timing of their consumption demand as a liquidity shock $\Longrightarrow$ a shock to preferences that change the relative price of marginal utility.
- The tension is between households aiming to maximize the return on their portfolio and minimize risk.
  1. Let $\lambda = 1$, households consume at $t = 1 \Longrightarrow$ demand for liquidity is large.
  2. Given $\lambda = 0$, there is no demand for liquidity because households consume at $t = 2$.
  3. When $\lambda \in (0, 1)$, the trade-off between return and risk aversion drives the demand for liquidity.

LIQUIDITY PREFERENCE, PORTFOLIO CHOICE, AND AUTARKY

- Suppose that at $t = 0$ a household splits its one unit of consumption good endowment between the short and long assets.
  1. $\theta \in (0, 1)$ is put into the short asset $\Longrightarrow C_1 = \theta$, and
  2. $1 - \theta$ goes to the long asset $\Longrightarrow C_2 = \theta + (1 - \theta)(1 + R)$.
- Since $C_1 < C_2$, $\lambda \mathcal{U}(C_1) + (1 - \lambda)\mathcal{U}(C_2)$ is lower compared with $\mathbf{E}\Big\{\mathcal{V}\Big(\lambda C_1 + (1 - \lambda)C_2\Big)\Big\}$.
- Household sets $\theta = 1$ to lower risk, but also lowers expected utility.
- A household solves $\text{Max}_\theta \Big[\lambda \mathcal{U}\Big(\theta\Big) + (1 - \lambda)\mathcal{U}\Big(\theta + (1 - \theta)(1 + R)\Big)\Big]$.
  1. FONC: $\lambda \mathcal{U}'\Big(\theta\Big) - (1 - \lambda)R\mathcal{U}'\Big(\theta + (1 - \theta)(1 + R)\Big) = 0$.
  2. Let $\mathcal{U}(C) = \ln C \Longrightarrow \theta = \dfrac{\lambda(1 + R)}{R}$; for $\theta \in (0, 1)$, $\dfrac{\lambda}{1 - \lambda} < R$.
  3. What is impact of $\mathcal{U}(C) = \dfrac{C^{1-\alpha} - 1}{1 - \alpha}$ on θ, $0 < \alpha$?

LIQUIDITY PREFERENCE, PORTFOLIO CHOICE, AND FINANCIAL INTERMEDIATION

- Suppose there are a countable infinite number of households each taking an address ℓ on the unit interval.

- If the households band together to form a financial intermediary,
  1. it takes their endowments as deposits, and
  2. invests in the short- and long-term assets.

- A social planning-FI exploits a law of large numbers.
  1. Withdrawals are λC_1 and $(1-\lambda)C_2$ on average at dates 1 and 2.
  2. The short- and long-term technologies constrain the FI to offer $C_1 = \dfrac{\theta}{\lambda}$ and $C_2 = \dfrac{1-\theta}{1-\lambda}(1+R)$ to depositors.

- This planner solves $\text{Max}_{\theta}\left[\lambda\mathcal{U}\left(\dfrac{\theta}{\lambda}\right) + (1-\lambda)\mathcal{U}\left(\dfrac{1-\theta}{1-\lambda}(1+R)\right)\right]$.
  1. FONC: $\mathcal{U}'\left(C_1\right) = (1+R)\mathcal{U}'\left(C_2\right) \Longrightarrow$ early and late consumers share risk up to the wedge $1+R \Longrightarrow C_1 < C_2$.
  2. Let $\mathcal{U}(C) = \ln C \Longrightarrow \theta = \lambda$ or complete risk sharing is efficient.

Optimal and Efficient Portfolio Choices

- Complete risk sharing is not efficient under non-logarithmic utility.
- The FI commits to returns that satisfy the average household.
  1. An early consuming household wants $C_1 = \lambda^{-1} > 1$ and less C_2.
  2. A late consumer desires $C_2 = \dfrac{1+R}{1-\lambda} > 1 + R$ and less at date 1.
  3. Given risk averse households, the FI will offer deposit contracts with date 1 and 2 payoffs somewhere in between $C_1 = \left(0,\, \lambda^{-1}\right]$ and $C_2 = \left(0,\, \dfrac{1+R}{1-\lambda}\right]$.
  4. $\Longrightarrow$ Tangency of intertemporal budget constraint and the highest indifference curve of the FI.
  5. The FI aggregates the utility functions of the early and late consuming households with λ and $1-\lambda$ serving as the Negishi weights.

Optimality, Efficiency, and the Diamond–Dybvig Model

- Note that a FI can promise $C_2 \in (0, 1+R]$, but $C_2 = 1+R$ if and only $C_1 \in (0, 1] \Longrightarrow C_1 \leq C_2$ is incentive efficient.
- Thus, add the constraint $C_1 \leq 1$ to a FI's optimization problem.
- A FI's problem is $\text{Max}_{C_1} \left[\lambda \mathcal{U}\left(C_1\right) + (1-\lambda)\mathcal{U}\left(\left[\frac{1+R}{1-\lambda}\right]\left(1-\lambda C_1\right)\right)\right]$, s.t. $C_1 \leq 1$.
- The FON/optimality condition is $\mathcal{U}'\left(C_1\right) - \frac{\xi}{\lambda} = (1+R)\mathcal{U}'\left(C_2\right)$, where ξ is the Lagrange multiplier on $C_1 \leq 1 \Longrightarrow \xi > 0$ if $C_1 = 1$.
- When $C_1 = 1$, $C_2 = 1+R \Longrightarrow$ efficient and first best solution.
- If $C_1 < 1$, $\mathcal{U}'\left(C_1\right) = (1+R)\mathcal{U}'\left(C_2\right) \Longrightarrow C_1 < C_2$, which is optimal and satisfies incentive constraint (*i.e.*, incentive efficient) $\Longrightarrow$ first best solution but there are many of these allocations.
- In either case, households have no reason to run the FIs.

Financial Frictions, Diversification, and Incomplete Markets

- The previous examples, Williamson (JPE, 1987), and BGG
  1. solve the maturity mismatch problem by exploiting
  2. versions of the law of large numbers that allow FIs to diversify their portfolios and insure depositors.
  3. The maturity mismatch is the financial friction.

- There are several reasons to be skeptical about this approach to financial markets in macro models.
  1. There are no markets selling AD securities at $t = 0$ that payoff at $t = 1, 2, 3, \ldots \Longrightarrow$ markets are incomplete.
  2. FIs lack unlimited resources $\Longrightarrow$ markets are incomplete.
  3. Insurance is costly for FIs to provide when markets are incomplete $\Longrightarrow$ the supply of insurance is limited.
  4. In an incomplete market, the price of insurance differs from its actuarially fair, complete market setting $\Longrightarrow \gtreqless$ is drive by risk aversion of buyers and sellers.

Liquidity Shocks and FIs

- This suggests another definition of liquidity $\Longrightarrow$ liquid assets have prices that are insensitive to changes in supply.
- Suppose that owners of the long asset can dump it at a loss at $t = 1$.
  1. At liquidation in $t = 1$, the long asset returns $\mathcal{R} < 1$.
  2. The loss on the long asset is $R - \mathcal{R}$ per unit invested.
- Assume FIs invest X_1 in the short asset and X_2 in the long asset at $t = 0$, where $X_1 + X_2 = 1$.
  1. When faced with depositors experiencing an unexpected liquidity shock at $t = 1$, FIs are forced to liquidate some of their holdings of the long asset $\Longrightarrow X_1 + \mathcal{R}X_2 < 1$.
  2. If $\mathbf{E}\{C_1\} > X_1 + \mathcal{R}X_2$, FIs fail.

Liquidity Shocks and REE Runs on FIs

- Do households anticipate a FI holding their deposits will default?
  1. When households expect other households to withdraw deposits, the rational response is to withdraw as well.
  2. $\Longrightarrow$ "Runs" on FIs are a rational expectations equilibrium (REE).
  3. Another REE has households not queuing to withdraw deposits.
  4. When depositors suffer liquidity shocks, multiple equilibria can result from FIs solving a maturity mismatch problem.
  5. Still, the implicit assumption is that households believe FIs lack the resources to satisfy the demand of depositors at once.

- The bank runs-REE can be eliminated
  1. by suspending convertibility prior to λ depositors claiming their funds from a FI $\Longrightarrow$ date 2 consumers expect the FI will have resources to pay their claims and do not run the FI,
  2. or with regulations that allow FIs to pay only a fraction of their deposits after a FI sees λ households withdraw $\Longrightarrow$ fractional-backing of deposits.

The Sequential Service Constraint and Bank Runs

- Suppose a FI does not know λ or is unaware when it is being run? $\Longrightarrow$ a household's liquidity shock is private information.
- These ideas are implicit in the notion of the "sequential service constraint" of Diamond and Dybvig (JPE, 1983).
  1. A sequential service constraint $\Longrightarrow$ households form a queue to withdraw deposits from a FI.
  2. The FI returns deposits to households first come first serve until it exhausts its assets.
- Suspending convertibility does not prevent households from believing in runs because the FI fails to realize it is being run.
- Draw λ from a distribution with observed moments, but regulators do not know whether to invoke fraction-backing of deposits or not.
- Depositors continue to expect runs can occur.

A Diamond–Dybvig Model: The Equilibrium Concept

- Allen and Gale (chapter 3.6) develop a Diamond and Dybvig (JPE, 1983) model of bank runs.

- The liquidity preference of households is still private information.
  1. For a SP-FI, truth telling by households is necessary to obtain an incentive compatible allocation that is also first best.
  2. Incentive compatibility $\Leftrightarrow C_1 \leq C_2$, given preferences and technology $\Longrightarrow$ date 2 households consume only at $t = 2$.
  3. A SP-FI facing $C_1 \leq C_2$ produces an incentive efficient allocation.
  4. If $C_1 < C_2$, the allocation is first best.

- These ideas inform our study of the Diamond and Dybvig model, but we depart from the original by employing a Nash equilibrium.
  1. At $t = 0$, a household's best policy rule is conditional on the expected actions of other households and FIs.
  2. Similarly, a FI's best date 0 policy rule is conditional on the expected actions of other FIs and households.

A Diamond–Dybvig Model: Extrinsic Uncertainty

- A bank run-REE is generated by beliefs about the actions that other economic agents might take.
- REEs not associated with economic or intrinsic fundamentals are generated by extrinsic beliefs about states of the world.
  1. Example: If a Canadian-based NHL team wins the Stanley Cup, there will be an economic boom in North America.
  2. Tying the state of the world to the outcome of an event that has no impact on the state is an example of a focal point.
  3. This is an example of an equilibrium generated by extrinsic uncertainty, which is sometimes called a sunspot equilibrium.
  4. Sunspots are about multiple equilibria.
- Suppose there is an event on which households focus or coordinate that when the event is observed households run FIs.
- Assume $\pi \in (0, 1)$ is the probability of the extrinsic event.

A Diamond–Dybvig Model: Set Up

- At $t = 0$, a FI, which insures households against unexpected liquidity shocks by pooling their deposits to investment in the short and long assets, only knows the following.
  1. With probability π, there is run and households that are early enough in the queue receive $X_1 + \mathcal{R}X_2$.
  2. The outcome is no run with probability $(1 - \pi)\lambda$, in which only early consuming households deposit in trade for C_1 at $t = 1$.
  3. There is another no run outcome in which only late consuming households deposit to receive C_2 at $t = 2$.

- The FI sees the representative household's lifetime preferences as

$$\mathbf{E}\left\{\mathcal{V}\left(X_1, X_2\right)\right\} = \pi\mathcal{U}\left(X_1 + \mathcal{R}X_2\right) + (1 - \pi)\left[\lambda\mathcal{U}\left(C_1\right) + (1 - \lambda)\mathcal{U}\left(C_2\right)\right],$$

to account for the probability of a run.

A Diamond–Dybvig Model: Optimality

- Given the possibility of a run, the FI's problem is

$$\text{Max}_{X_1}\, \pi\, \mathcal{U}\Big(X_1 + \mathcal{R}(1 - X_1)\Big)$$

$$+ \ (1-\pi)\left[\lambda\,\mathcal{U}\left(\frac{X_1}{\lambda}\right) + (1-\lambda)\,\mathcal{U}\left(\frac{1-X_1}{1-\lambda}\Big(1+R\Big)\right)\right].$$

- The FON/optimality condition is

$$\pi(1-\mathcal{R})\,\mathcal{U}'\Big(X_1 + \mathcal{R}X_2\Big) + (1-\pi)\,\mathcal{U}'\Big(C_1\Big) = (1-\pi)(1+R)\,\mathcal{U}'\Big(C_2\Big).$$

- The FI holds less of the long asset to hedge against the probability of a run $\Longrightarrow$ the MU of C_2 is higher to compensate for the need to transfer utility to households that run.

A DIAMOND–DYBVIG MODEL: MOTIVATING BANK RUNS

- The first-best solution results
  1. if $\pi = 0$, $\mathcal{U}'\left(C_1\right) = (1 + R)\mathcal{U}'\left(C_2\right)$, or
  2. if the coefficient of RRA, $\mathscr{C} = -\dfrac{\mathcal{U}''\left(C\right)}{\mathcal{U}'\left(C\right)}C \leq 1$,
  3. which has the FI setting $0 < C_1 \leq C_2 \Longrightarrow$ REE without bank runs.
- RE runs equilibria rely on greater risk aversion, $\mathscr{C} > 1$.
- Also, assume $\mathcal{R} = 1 \Longrightarrow$ since long and short assets yield equivalent returns at $t = 1$, FIs only hold the long asset.
- However, FIs credibly commit to $\left(C_1, C_2\right) = \left(1, 1\right) \Longrightarrow t = 2$ consumers have no reason to run.
- Expectations of a extrinsic event coordinates late consuming households to withdraw deposits at $t = 1 \Longrightarrow$ run a FI.
- Bank runs are probabilistic (uncertain) events $\Longrightarrow$ are bank runs predictable?

A Diamond–Dybvig Model: Optimal Contracts

- A FI aims to $\text{Max}_{\{C_1,\, C_2\}} \lambda \mathcal{U}\big(C_1\big) + (1-\lambda)\mathcal{U}\big(C_2\big)$, in a non-first best world, which is s.t. $(1+R)\lambda C_1 + (1-\lambda)C_2 \leq 1 + R$.

- The non-first best world has the potential for runs on FIs.

- The budget constraint is the result of balancing $(1+R)(1-\lambda C_1)$ against $(1-\lambda)C_2$.
  1. A FI finds $1 - \lambda C_1$ units of the asset left on its balance sheet
  2. after being forced to liquidate λC_1 units of its assets
  3. in response to the demand of date 1 consumers.

- Since the FI cannot credible commit to $(1+R)(1-\lambda C_1) < (1-\lambda)C_2$ at date 0, the FI can at most promise $1 + R$ units of consumption at date 2 per unit of date 1 consumption.

A Diamond–Dybvig Model: Optimal Contracts sans Bank Runs

- A FI aims to $\text{Max}_{\{C_1,\, C_2\}} \lambda \mathcal{U}\big(C_1\big) + (1-\lambda)\mathcal{U}\big(C_2\big)$, to achieve the first best, which is s.t. $(1+R)\lambda C_1 + (1-\lambda)C_2 \leq 1 + R$ and $C_1 \leq 1$.
- We have already studied the FI's optimization problem when the solution is restricted to be incentive efficient.
- The efficient or first best world solution is $\big(C_1,\, C_2\big) = \big(1,\, 1+R\big)$.
- The efficient-no bank runs equilibrium rests on the FI credibly committing to the second constraint $C_1 \leq 1$.

A Diamond–Dybvig Model: FI Run and No-Run Investment Strategies

- If $\mathscr{C} > 1$, $\mathcal{R}$, and $\pi \in (0, 1)$, the FI can follow two different investment strategies.
- The FI does not commit to $C_1 \leq 1$, which results in
  1. an investment only in the short asset given π and
  2. at probability $1 - \pi$ holding only the long asset.
- A commitment is made by the FI to $C_1 \leq 1$, which yields $\big(C_1, C_2\big) = \big(1, 1 + R\big)$.
- Household expected utility $= \lambda\mathcal{U}\big(1\big) + (1 - \lambda)\mathcal{U}\big(1 + R\big)$ in the no-run equilibrium.

A Diamond–Dybvig Model: The Runs REE

- The bank run outcome produces defaults by FIs and $\left(C_1, C_2\right) = \left(1, 0\right)$ with probability π.

- The no-bank run outcome occurs at rate $1 - \pi$ and call the deposit outcome-consumption bundle $\left(C_1, C_2\right) = \left(\tilde{C}_1, \tilde{C}_2\right)$.

- Given the rate of bank runs is π, household expected utility is $\pi\mathcal{U}\left(1\right) + (1-\pi)\left[\lambda\mathcal{U}\left(\tilde{C}_1\right) + (1-\lambda)\mathcal{U}\left(\tilde{C}_2\right)\right]$.

A DIAMOND–DYBVIG MODEL: MULTIPLE REE

- The FI chooses between the two strategies by comparing expected household utility across the bank runs and no-bank runs REE.

- Households expect to be better off running FIs at rate π if

$$\pi\mathcal{U}\big(1\big) + (1-\pi)\Big[\lambda\mathcal{U}\big(\tilde{C}_1\big) + (1-\lambda)\mathcal{U}\big(\tilde{C}_2\big)\Big] > \lambda\mathcal{U}\big(1\big) + (1-\lambda)\mathcal{U}\big(1+R\big).$$

- The expected utility of the no-run REE dominates the certain utility under a bank run but is dominated by the certain utility of the no-run allocation that is not reduced by the constraint $C_1 \leq 1$, or

$$\mathcal{U}\big(1\big) < \lambda\mathcal{U}\big(1\big) + (1-\lambda)\mathcal{U}\big(1+R\big) < \lambda\mathcal{U}\big(\tilde{C}_1\big) + (1-\lambda)\mathcal{U}\big(\tilde{C}_2\big).$$

- The implication is there is a $\overline{\pi} \in \big(0,\, 1\big)$ that

$$\overline{\pi}\mathcal{U}\big(1\big) + (1-\overline{\pi})\Big[\lambda\mathcal{U}\big(\tilde{C}_1\big) + (1-\lambda)\mathcal{U}\big(\tilde{C}_2\big)\Big] = \lambda\mathcal{U}\big(1\big) + (1-\lambda)\mathcal{U}\big(1+R\big).$$

$\Longrightarrow$ FIs are indifferent between the runs-REE and the no-run REE.

A Diamond–Dybvig Model: Summary

- Another implication of the bank runs model with multiple REE is that $\overline{\pi}$ is relatively small.

- As $\overline{\pi} \longrightarrow 1$, $\mathcal{U}\left(1\right) < \mathcal{U}\left(1 + R\right) \Longrightarrow$ late consuming households are better off waiting to $t = 2$ as the probability of a bank run increases.

- A FI raises the welfare of early and late consuming households by
  1. offering deposit contracts predicted on the constraint $C_1 \leq 1$ to rule out the bank runs-REE,
  2. given the extrinsic probability of a bank run is high.

- Nash equilibrium concept drives analysis of the Diamond–Dybvig model.
  1. Different equilibrium selection devices alter the incentives facing FIs
  2. and households, but runs will remain an equilibrium.

The Diamond–Dybvig Model: Other Thoughts

- The Diamond–Dybvig model comes with an important caveat.
  1. As Diamond and Dybvig (JPE, 1983) note, they ignore whether a FI faces incentives to engage in moral hazard on the asset side of its balance sheet.
  2. Will FIs hold riskier assets because there is insurance against liquidity shocks (*i.e.*, government-backed deposit insurance)?
- FIs are able to pool the risk of liquidity shocks because households are not allowed to trade subsequent to their liquidity being realized.
  1. Unrealized arbitrage opportunities exist in Diamond–Dybvig models.
  2. FIs are transferring resources to (*i.e.*, subsidizing) early consuming households from late (*i.e.*, $t = 2$) consumers.
- Closing these financial markets has social costs because of efficiency losses $\Longrightarrow$ wedges exist between rates in money and credit markets.
- Are efficiency gains from opening these markets outweighed by increased financial instability?

Introduction

- The BGG and KM classes of DSGE models focus on the banking accelerator.
  1. *Financial Accelerator*: expansionary monetary policy shock increases the MP_K, which raises the collateral value of capital loosening financing constraints facing entrepreneurs.
  2. $\Longrightarrow$ reduces the external finance premium.

- Gertler and Kiyotaki (2013, NBER wp–19129) adapt the Gertler and Karadi (JME, 2011) DSGE model $\Longrightarrow$ KM style model with households and banks.
  1. Households are less productive than banks at running projects, but capital constrained banks need household deposits to fund projects.
  2. GK study the roles these financial frictions have in creating bank runs.

Gertler and Kiyotaki (2013, NBER wp–19129): Capital Markets

- Capital is held by households, $K_{H,t}$, and banks, $K_{B,t}$, and sum to unity, $K_{H,t} + K_{B,t} = 1$ (*i.e.*, a normalization).

- Capital produces the single nondurable good that households consume.
  1. Label the productivity shock Z_{t+1}, which is common to banks and households.
  2. The banking sector maps $K_{B,t}$ into $Z_{t+1}K_{B,t}$ units of the nondurable good.
  3. When households produce $Z_{t+1}K_{H,t}$ units of the nondurable good, they generate costs of $f(K_{H,t}) \Longrightarrow$ households are less competent than banks at managing projects.

- Household "management" costs are a nonconvex function of $K_{H,t}$ and $\overline{K}_{H,t} \in (0, 1)$,

$$f(K_{H,t}) = \begin{cases} \dfrac{\alpha}{2}K_{H,t}^2, \ \ 0 < \alpha, & \text{for } K_{H,t} \leq \overline{K}_{H,t}, \\[2ex] \alpha\overline{K}_{H,t}\left(K_{H,t} - \dfrac{\overline{K}_{H,t}}{2}\right), & \text{for } K_{H,t} > \overline{K}_{H,t}. \end{cases}$$

- Costs are quadratic (and convex) in $K_{H,t}$ up to $\overline{K}_{H,t}$ and linear in $K_{H,t}$ thereafter $\Longrightarrow$ if FIs shut down, households will absorb the entire capital stock to produce.

- Assume $K_{H,t}$ and $K_{B,t}$ do not depreciate.

GERTLER AND KIYOTAKI (2013, NBER WP–19129): HOUSEHOLDS

- Households consume, $C_{H,t}$ and save either $K_{H,t}$ or by depositing, D_t, with banks.
- Preferences, $\mathcal{U}_t = \mathbf{E}_t\left\{\sum_{j=0}^{\infty}\beta^{t+j}\ln C_{H,t+j}\right\}$, are maximized by households s.t.

$$C_{H,t} + D_t + Q_t K_{H,t} + f(K_{H,t}) = Z_t \mathcal{W} + R_t D_{t-1} + (Z_t + Q_t)K_{H,t-1},$$

  where Q_t, $\mathcal{W}$, and R_t denote the price of capital, "labor" income, and the date t return to date $t-1$ deposits $\Longrightarrow$ time deposits that are one-period debt contracts issued by banks that pay R_t at date t.

- There is no liquidity preference shock driving household deposit decisions $\Longrightarrow$ GK assume that bank runs are never expected by households.
- If a bank run occurs, households face a sequential service constraint.
  1. Households may or may not receive the deposit in full given a bank run.
  2. Given a bank run and a household is low in the queue, the household buys $K_{B,t}$.
  3. $\Longrightarrow$ Otherwise, banks have no resources to pay off their liabilities to households.
  4. Lacking a labor supply decision, household consumption-saving decisions take the brunt of a bank run $\Longrightarrow$ in a run $\overline{K}_{H,t} < K_{H,t}$ $\Longrightarrow$ a fire sale drives Q_t low.

Gertler and Kiyotaki (2013, NBER wp–19129): Household FONCs

- With respect to $C_{H,t},\ D_t$, and $K_{H,t}$, the FONCs are

$$\begin{aligned} \beta^t C_{H,t}^{-1} &= \lambda_t, \\ \lambda_t &= \mathbf{E}_t\Big\{R_{t+1}\lambda_{t+1}\Big\}, \\ \Big[1 + \frac{f'(K_{H,t})}{Q_t}\Big]\lambda_t &= \mathbf{E}_t\left\{\left[\frac{Q_{t+1}}{Q_t}\left(1 + \frac{Z_{t+1}}{Q_{t+1}}\right)\right]\lambda_{t+1}\right\}, \end{aligned}$$

  where λ_t is a Lagrange multiplier tied to the household's budget constraint.

- These have standard interpretations except for the FONC w/r/t to $K_{H,t}$.
  1. When the household increases its capital stock by a unit,
  2. the cost equals the utility loss of a unit of consumption plus the cost of operating the project.
  3. These costs are balanced against the discounted expected benefits of the one unit increase in the household capital stock, which are
  4. "capital gains," Q_{t+1}/Q_t, scaled by that one unit of capital plus its value in production, Z_{t+1}/Q_{t+1}.

GERTLER AND KIYOTAKI (2013, NBER WP–19129): HOUSEHOLD OPTIMALITY

- Let $\Lambda_{t,t+1} \equiv \dfrac{\lambda_{t+1}}{\lambda_t} = \beta\left(\dfrac{C_{H,t+1}}{C_{H,t}}\right)^{-1}$, the FONCs yield two optimality conditions

$$1 = \mathbf{E}_t\left\{\Lambda_{t,t+1}R_{t+1}\right\},$$

$$1 = \mathbf{E}_t\left\{\Lambda_{t,t+1}R_{H,t+1}\right\}, \quad R_{H,t+1} = \frac{Q_{t+1}}{Q_t}\left[1+\frac{Z_{t+1}}{Q_{t+1}}\right]\Big/\left[1+\frac{f'(K_{H,t})}{Q_t}\right]$$

- When $K_{H,t} > 0$, the Euler equation of $R_{H,t+1}$ is an implicit capital demand equation.
  1. The household's "capital demand function" is restricted by $f'(K_{H,t}) = \alpha K_{H,t}$, given $K_{H,t} \in (0, \overline{K}_{H,t})$ or $f'(K_{H,t}) = \alpha\overline{K}_{H,t}$, given $K_{H,t} \in [\overline{K}_{H,t}, 1]$.
  2. On the range $(0, \overline{K}_{H,t})$, $f'(K_{H,t}) \uparrow \Longrightarrow Q_t$ should be strictly decreasing as $K_{H,t}$ rises toward one, where $K_{H,t} \in (0, 1]$.

- The Euler equations also restrict arbitrage between the markets for capital and D_t

$$\mathbf{E}_t\left\{\Lambda_{t,t+1}\left(R_{H,t+1}-R_{t+1}\right)\right\} = 0 \Longrightarrow \mathbf{E}_t\left\{R_{H,t+1}-R_{t+1}\right\} = -\frac{\mathrm{Cov}\left(R_{H,t+1}-R_{t+1},\, \Lambda_{t,t+1}\right)}{\mathbf{E}_t\Lambda_{t,t+1}}.$$

- $\Longrightarrow$ If the inverse of the consumption growth rate and the interest rate spread on $K_{H,t}$ over D_t is negative, the spread is large in states of the world with low Z_t.

GERTLER AND KIYOTAKI (2013, NBER WP–19129): BANKS

- Banks are risk neutral, finite lived, their asset is $K_{B,t}$ and they issue liabilities in the form of deposits and equity or net worth, N_t.
- The financial friction is moral hazard $\Longrightarrow$ a bank can abscond with a fraction θ of the market value of its capital, $Q_t K_{B,t}$.
  1. If a bank engages in fraud at date t, its depositors close the bank at the beginning of date $t+1$.
  2. A bank decides whether to "take the money" and suffer a shutdown using a benefit-cost calculation.
  3. Let $\mathcal{V}_t$ = EPDV of consumption when not taking $\theta Q_t K_{B,t} \Longrightarrow \theta Q_t K_{B,t} \leq \mathcal{V}_t$.
  4. Without financial frictions, $K_{B,t} = 1 \Longrightarrow$ assume the ICC binds $\theta Q_t K_{B,t} = \mathcal{V}_t$.
  5. $\Longrightarrow$ Study the representative bank.
- Banks also exit because they last from t to $t+1$ at *iid* probability $\sigma \in (0, 1) \Longrightarrow$ pass on at probability $1 - \sigma \Longrightarrow$ banks' expected life = $1/(1-\sigma)$.
  1. A living bank's preferences are $\mathcal{V}_t = \mathbf{E}_t \sum_{j=1}^{\infty} \beta^j (1-\sigma)\sigma^{j-1} C_{B,t+j}$.
  2. The probability a bank exits at $t+j$ is $(1-\sigma)\sigma^{j-1}$ and $C_{B,t+j}$ is its consumption at that date, which is N_{t+j}.
- A new bank is endowed with equity of ω_B while existing banks accumulate net worth only from retained earnings, $N_t = (Z_t + Q_t)K_{B,t-1} - R_t D_{t-1}$.
- $\Longrightarrow$ Since living banks cannot issue new equity issue, the return on their assets net of liabilities is their only source for "growing" their balance sheet $N_t = Q_t K_{B,t} - D_t$.

GERTLER AND KIYOTAKI (2013, NBER WP–19129): THE BANK'S PROBLEM, I

- The bank's preferences are recursive $\mathcal{V}_t = \beta \mathbf{E}_t \Big\{ (1-\sigma) N_{t+1} + \sigma \mathcal{V}_{t+1} \Big\}$.
- The dynamic program of the bank is to maximize $\mathcal{V}_t$ by choosing its capital, $K_{B,t}$, and deposit contract, D_t, s.t. the ICC, retained earnings, and balance sheet constraints.
  1. Conjecture the solution is linear in capital and deposits,

$$\begin{aligned} \mathcal{V}_t &= \nu_{K,t} K_{B,t} - \nu_{D,t} D_t = \frac{\nu_{K,t}}{Q_t} Q_t K_{B,t} - \nu_{D,t} D_t \\ &= \left(\frac{\nu_{K,t}}{Q_t} - \nu_{D,t} \right) Q_t K_{B,t} + \nu_{D,t} \left(Q_t K_{B,t} - D_t \right) \\ &= \mu_t Q_t K_{B,t} + \nu_{D,t} N_t, \quad \text{where} \quad \mu_t = \frac{\nu_{K,t}}{Q_t} - \nu_{D,t}. \end{aligned}$$

  2. $\Longrightarrow \mu_t$ is the marginal value of a unit of capital to the PDV of banker utility, which is net the marginal cost of accepting one more unit of deposits.
- The ICC becomes $(\theta - \mu_t) Q_t K_{B,t} = \nu_{D,t} N_t \Longrightarrow \mu_t \in (0, \theta)$ for $N_t > 0$.
  1. Define the bank's leverage ratio $\phi_t \equiv \dfrac{Q_t K_{B,t}}{N_t} = \dfrac{\nu_{D,t}}{\theta - \mu_t}$.
  2. Bank leverage is limited by $\theta - \mu_t > 0 \Longrightarrow$ the bank scales up its balance sheet with risky projects until the cost of losing the PDV of its future terminal consumption equals the benefits of committing fraud on its depositors.

Gertler and Kiyotaki (2013, NBER wp–19129): The Bank's Problem, II

- Return to $\mathcal{V}_{t+1} = \mu_{t+1}Q_{t+1}K_{B,t+1} + \nu_{D,t+1}N_{t+1}$, $Q_{t+1}K_{B,t+1} = \phi_{t+1}N_{t+1}$, $N_{t+1} = (Z_{t+1} + Q_{t+1})K_{B,t} - R_{t+1}D_t$, and $D_t = Q_tK_{B,t} - N_t$

$$\begin{aligned}\mathcal{V}_{t+1} &= (\mu_{t+1}\phi_{t+1} + \nu_{D,t+1})N_{t+1}\\ &= (\mu_{t+1}\phi_{t+1} + \nu_{D,t+1})(Z_{t+1} + Q_{t+1})K_{B,t} - R_{t+1}D_t\\ &= (\mu_{t+1}\phi_{t+1} + \nu_{D,t+1})(Z_{t+1} + Q_{t+1})K_{B,t} - R_{t+1}(Q_tK_{B,t} - N_t)\\ &= (\mu_{t+1}\phi_{t+1} + \nu_{D,t+1})(R_{B,t+1} - R_{t+1})Q_tK_{B,t} + R_{t+1}N_t,\ R_{B,t} = (Z_t + Q_t)/Q_{t-1}.\end{aligned}$$

- Then $\mathcal{V}_t = \beta\mathbf{E}_t\Big\{(1-\sigma)N_{t+1} + \sigma\mathcal{V}_{t+1}\Big\}$ becomes

$$\mu_tQ_tK_{B,t} + \nu_{D,t}N_t = \beta\mathbf{E}_t\Big\{\Big[1 + \sigma(\mu_{t+1}\phi_{t+1} + \nu_{D,t+1} - 1)\Big]\Big[(R_{B,t+1} - R_{t+1})Q_tK_{B,t} + R_{t+1}N_t\Big]\Big\}.$$

- Remember $\mathcal{V}_t = \mu_tQ_tK_{B,t} - \nu_{D,t}D_t \Longrightarrow \nu_{D,t} = \beta R_{t+1}\mathbf{E}_t\Big\{\Big[1 + \sigma(\mu_{t+1}\phi_{t+1} + \nu_{D,t+1} - 1)\Big]\Big\}$ and $\mu_t = \beta\mathbf{E}_t\Big\{\Big[1 + \sigma(\mu_{t+1}\phi_{t+1} + \nu_{D,t+1} - 1)\Big](R_{B,t+1} - R_{t+1})\Big\}$.
  1. $\nu_{D,t}$ and μ_t form a bivariate system of forward looking first-order stochastic difference equations $\Longrightarrow$ solve taking ϕ_{t+1}, $R_{B,t+1}$, and R_{t+1} as given processes.
  2. R_t discounts $\nu_{D,t}$ while the spread on the bank's return to capital discounts μ_t.
  3. $\mathbf{E}_t\Big\{\Big[1 - \sigma + \sigma(\mu_{t+1}\phi_{t+1} + \nu_{D,t+1})\Big]\Big\}$ = expected value of the bank.
  4. A living bank values one more unit of net worth because it raises $\mathcal{V}_t$ by $\nu_{D,t}$ $\Longrightarrow$ owe households less deposits plus the return.
  5. Allows the bank to become more leveraged, which raises $\mathcal{V}_t$ by $\mu_t\phi_t$.

Gertler and Kiyotaki (2013, NBER wp–19129): Bank Runs, I

- Bank runs are driven by fundamentals not sunspots.
- Households decide to run the bank if the value of the bank's assets in liquidation are less than its liabilities $\Longrightarrow (Q_t^* + Z_t)K_{B,t-1} < R_t D_{t-1}$, where Q_t^* is the market price of capital when the bank is shut down.
- Since $D_{t-1} = Q_{t-1}K_{B,t-1} - N_{t-1} \Longrightarrow (Q_t^* + Z_t - R_t Q_{t-1})K_{B,t-1} < -R_t N_{t-1}$ or $\left[\frac{Q_t^* + Z_t}{Q_{t-1}} - R_t\right] Q_{t-1}K_{B,t-1} < -R_t N_t$ and using the definition of ϕ_{t-1}

$$\left[R_{B,t}^* + R_t\right]\phi_{t-1} < R_t \Longrightarrow \frac{R_{B,t}^*}{R_t} < 1 - \frac{1}{\phi_{t-1}},$$

- When the spread of the liquidation value of bank assets over the deposit rate is small and/or bank leverage is high, households run the bank.
- Bank runs are state dependent in the GK model.
  1. In this economy, ϕ_t is the state of the aggregate credit market.
  2. However, the liquidation price of capital Q_t^* is not a function of ϕ_t.

Gertler and Kiyotaki (2013, NBER wp–19129), Bank Runs, II

- In the GK model, households are a sink absorbing all $K_{B,t}$ in a run $\Longrightarrow f(\cdot)$ is an incentive for households to accumulate capital when its price is falling.
- When the bank is run, its entire capital stock is sold to households, $K_{H,t} = 1$
  1. $\Longrightarrow C_{H,t} = 1 + Z_t \mathcal{W} - f(1)$.
  2. When the bank has closed, the household Euler equation for capital becomes $1 = \mathbf{E}_t\left\{\Lambda_{t,t+1} R^*_{H,t+1}\right\}$, where $R^*_{H,t+1} = (Q^*_{t+1} + Z_{t+1})/(Q^*_t + \alpha\overline{K}_H)$.
  3. The household holds the entire capital stock the marginal cost of operating a project is at the fixed kink point $\overline{K}_H$ when $K_{H,t} = 1$.
- Iterating the Euler equation $Q^*_t + \alpha\overline{K}_H = \mathbf{E}_t\left\{\Lambda_{t,t+1}\left(Q^*_{t+1} + Z_{t+1}\right)\right\}$ produces

$$Q^*_t = \mathbf{E}_t\left\{\sum_{j=1}^{\infty} \Lambda_{t,t+1}\left(Z_{t+j} - \alpha\overline{K}_H\right)\right\} - \alpha\overline{K}_H.$$

- The liquidation price is the expected present discounted value of productivity shocks $\Longrightarrow$ a persistent sequence of bad productivity shocks generate bank runs by lowering the liquidation price of and return on bank capital.
- Z_t is an aggregate shock in this economy, which cannot be diversified away.
  1. Perhaps fiscal and/or monetary policymakers can solve this bank run problem.
  2. $\Longrightarrow$ Deposit insurance, lender of last resort authority, capital requirements, and/or limits on leverage?

Introduction: Kiyotaki and Moore (IER, 2005)

- Observation: Money is held by households, but is dominated in rate of return by most other assets.
- KM (2005) want to study this quandary in terms of the impact of liquid assets on the value of less liquid assets.
- A liquid asset is traded in a deep market.
  1. There is substantial supply and demand at all moments in time.
  2. Low costs to trade $\Longrightarrow$ potentially many owners before maturity.
- Claim: Liquid assets held because of value in trade rather than only for returns and principal.

Sketch of a DSGE model with Liquid and Illiquid Assets

- Questions: Do liquid assets matter for allocations and prices in DSGE models?
- However, liquid assets are denominated in units of the consumption good rather than in terms of a fiat currency.
- The DSGE model has physical capital and land as its assets.
  1. Capital, k_t, is accumulated through investment, i_t.
  2. The aggregate supply of land is fixed at $\overline{L}$.
  3. The economy lacks fiat currency.
- Only a fraction Π of households can invest at date t.
  1. Investors turn a unit of consumption, c_t, into one unit of k_t.
  2. The chance to invest is *iid* across time and households. $\Longrightarrow$ today's investors may not be able to invest tomorrow.
  3. The remaining $1 - \Pi$ measure of households trade with investors to obtain returns on capital, $r_{K,t}$, and land, $r_{L,t}$.

Sketch of a DSGE model with Liquid and Illiquid Assets, cont.

- Assume that households can sell only a fraction θ of their k_t during any date $t \Longrightarrow$ capital is illiquid.
- Households can be liquidity constrained.
  1. They may have insufficient k_t to run a project.
  2. Other households can lend only θ of their k_t
  3. $\Longrightarrow$ some investment opportunities may go unused.
- Assume land is a liquid asset, which serves to collateralizes investments in k_t.
- Liquidity constraints $\Leftrightarrow$ collateral constraints.
- Land is vital to achieve "efficient" allocations.
  1. The probability Π a household is tapped to invest is small $\forall\ t$.
  2. When a household can invest, its land is offered as collateral against many "loans" of capital from other households.

Sketch of a DSGE model with Liquid and Illiquid Assets, cont.

- If households expect collateral constraints to bind, land is in greater demand than otherwise $\Longrightarrow$ its liquidity services obtain a premium.
- The liquidity premium may be large enough for $r_{L,t} < r_{K,t}$.
  1. $\Longrightarrow$ Land's liquidity services is a reason to hold it in face of a greater return to k_t and possible for $r_{L,t} < \beta^{-1}$, $\beta \in (0, 1)$.
  2. $\Longrightarrow$ The rate of time preference dominates the return to land.
- Since $r_{L,t}$ moves inversely with the price of land, p_t, higher p_t signals greater liquidity in the market for land.
  1. Greater liquidity encourages more trade between investing and non-investing households.
  2. $\Longrightarrow$ Increasing the economy's aggregate capital stock and the potential for more output in the future.
- The interaction of liquidity and investment affects the persistence and volatility aggregate fluctuations, according to KM.

A DSGE model with Liquid and Illiquid Assets

- Unit mass of households with preferences: $V_t = \mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j \ln c_{t+j} \right\}$.
- Technology: $y_t = a_t \mathbb{k}_t^{\alpha} \ell_t^{1-\alpha}$, $\alpha \in (0, 1)$.
  1. a_t is TFP, evolves as stationary first-order Markov process, and is common to all households.
  2. y_t output of single consumption-capital good.
  3. $\mathbb{k}_t$ and ℓ_t are capital and land in production.
- Budget constraint:

$$c_t + i_t + (k_{t+1} - i_t - \lambda k_t)q_t + (m_{t+1} - m_t)p_t$$
$$= y_t - r_{K,t}(\mathbb{k}_t - k_t) - r_{L,t}(\ell_t - m_t),$$

where q_t and k_t (m_t) denote the price of installed capital, capital (land) held by the household prior to investment.

A DSGE model with Liquid and Illiquid Assets: Collateral Constraints

- The law of motion of capital is $k_{t+1} \geq (1 - \theta)(\lambda k_t + i_t)$, where $1 - \lambda$, $\lambda \in (0, 1)$ is the depreciation rate of k_t.
  1. After investment and depreciation, a household has $\lambda k_t + i_t$.
  2. The household has at least $1 - \theta \times \lambda k_t + i_t$ units of capital to take into next period.
  3. But a household can lend at most θ units of this capital.
  4. $\Longrightarrow$ Imposes a collateral constraint on an investing household.

- The short sale constraint on land is $0 \leq m_{t+1}$.
  1. A household cannot buy the rights to land in the future.
  2. Implicit restriction is there are no options markets to trade or insure against future investment opportunities.
  3. KM assume that a household has only a moment to decide when it is tapped for an investment opportunities.
  4. A household can only offer $p_t m_t$ as collateral a "land in advance constraint."

A DSGE model with Liquid and Illiquid Assets: Equilibrium Definition

- A competitive equilibrium consists of pricing functions $p_t = p(S_t)$, $q_t = q(S_t)$, $r_{K,t} = r_K(S_t)$, and $r_{L,t} = r_L(S_t)$ along with household choices of $\begin{bmatrix} c_t & i_t & k_t & \Bbbk_t & m_t & \ell_t \end{bmatrix}'$, such that
  1. utility V_t is maximized s.t. budget and collateral constraints,
  2. aggregate capital: $K_t = \int_0^1 k_{t,j} dj$,
  3. aggregate land: $\overline{L}_t = \int_0^1 m_{t,j} dj$,
  4. aggregate resource constraint: $Y_t = C_t + I_t$, $C_t = \int_0^1 c_{t,j} dj$, and $I_t = \int_0^1 i_{t,j} dj$, and
  5. the rental markets for capital and land clear at strictly positive $r_{K,t}$ and $r_{L,t}$,
  6. where the aggregate state is $S_t = \begin{bmatrix} K_t & a_t \end{bmatrix}'$.
- This is not a recursive equilibrium.

A DSGE model with Liquid and Illiquid Assets: Market Returns

- Since existing markets are perfectly competitive and an investing household operates the production technology to maximize profit, equilibrium profit is zero during every date t.

- Returns to factor inputs equal relevant marginal products:

$$r_{K,t} = \alpha a_t \left(\frac{\ell_t}{\mathbb{k}_t}\right)^{1-\alpha} = \alpha \frac{y_t}{\mathbb{k}_t} \text{ and } r_{L,t} = (1-\alpha) a_t \left(\frac{\mathbb{k}_t}{\ell_t}\right)^{\alpha} = (1-\alpha)\frac{y_t}{\ell_t}.$$

A DSGE model with Liquid and Illiquid Assets: Tobin's q

- Tobin's q is the ratio of the market value of a firm's capital to the cost of replacing that capital at market prices, but often measured as the ratio of the market value of firm's equity to its book value.
- If $q_t > 1$, a household invests given it has been tapped to do so.
  1. Otherwise, the value of $\mathbb{k}_t$ is less than the value of k_t
  2. $\Longrightarrow$ value of investing and producing $<$ consumption.
- The smaller is θ the tighter are the collateral constraints on non-investing households.
- Or given θ, the smaller is the capital owned by non-investor households the less they can rent to investors.
- The liquidity value of land also places limits on the amount of capital non-investors have available for the rental market.
- This is the non-recursive propagation mechanism of the KM model $\Longrightarrow$ quantities generate price movements and in return prices drive fluctuations in quantities.

A DSGE model with Liquid and Illiquid Assets: Budget Constraints

- A household not tapped to invest faces the budget constraint

$$c_{N,t} + q_t k_{N,t+1} + p_t m_{N,t+1} \;=\; (r_{K,t} + \lambda q_t)k_t + (r_{L,t} + p_t)m_t \;\equiv\; \mathcal{W}_{N,t},$$

because its $\Bbbk_t = \ell_t = 0$, where N denotes a non-investing household and $\mathcal{W}_{N,t}$ is the wealth of this household.

- If $q_t > 1$ and the collateral constraints bind, an investor faces

$$c_{I,t} + \frac{1 - \theta q_t}{1 - \theta} k_{I,t+1} + p_t m_{I,t+1} \;=\; (r_{K,t} + \lambda)k_t + (r_{L,t} + p_t)m_t \;\equiv\; \mathcal{W}_{I,t},$$

where I denotes an investing household and $\mathcal{W}_{I,t}$ is its wealth.

- The term $\dfrac{1 - \theta q_t}{1 - \theta}$ represents the investing household's "leverage."
  1. This household pays $1 - \theta q_t$ per unit of installed capital to run the project, but still rents θ of its capital to other investors.
  2. $\Longrightarrow 1 - \theta$ (per unit of capital) of the project remains on the asset side of investor's balance sheet.

A DSGE model with Liquid and Illiquid Assets: Equilibrium

- The Euler equations for investing and non-investing households are standard, except for the structure of this version of the KM model.
  1. Only Π of the households can invest during any date t.
  2. Household portfolios include k_t and m_t.
  3. Tobin's q is in force: $1 < q_t$.

- Euler equations imply consumption is a fixed fraction of wealth, given preferences are log over uncertain consumption streams.
  1. $c_{N,t} = (1-\beta)\mathcal{W}_{N,t}$ and $c_{I,t} = (1-\beta)\mathcal{W}_{I,t}$.
  2. Investors carry no land from today into tomorrow, $m_{I,t+1} = 0$, and $k_{t+1} = (1-\theta)(\lambda k_t + i_t)$.
  3. $\Longrightarrow c_{I,t} + (1-\theta q_t)i_t = (r_{K,t} + \theta\lambda q_t)k_t + (r_{L,t} + p_t)m_t$.
  4. Or $(1-\theta q_t)i_t = \mathcal{W}_{I,t} - c_{I,t} - (1-\theta)\lambda q_t k_t \Longrightarrow$ investing households finance their share of investment out of their own wealth net of their consumption and net of the value of the capital they own at the start of date t.

A DSGE model with Liquid and Illiquid Assets: Optimality for Non-Investors

- Non-investing households solve their maximization problem to find

$$\frac{q_t}{c_{N,t}} = \beta \mathbf{E}_t \left\{ \Pi \frac{r_{K,t+1} + \lambda}{c_{I,t+1|N,t}} + \left(1 - \Pi\right) \frac{r_{K,t+1} + \lambda q_{t+1}}{c_{N,t+1|N,t}} \right\},$$

and

$$\frac{p_t}{c_{N,t}} = \beta \mathbf{E}_t \left\{ \Pi \frac{r_{L,t+1} + p_{t+1}}{c_{I,t+1|N,t}} + \left(1 - \Pi\right) \frac{r_{L,t+1} + p_{t+1}}{c_{N,t+1|N,t}} \right\},$$

where $c_{j,t+1|N,t}$ is consumption at $t+1$ of a household engaged in j, $j = I, N$, at that time, conditional on its not investing at date t.

- A non-investing household can acquire a unit of k_{t+1} $(m_{N,t+1})$ at a cost of a unit of marginal utility valued at q_t (p_t).
- This cost is balanced against the discounted expected benefit which is a weighted average of date $t+1$ marginal utility
  1. of investors valued at their gross return to k_{t+1} $(m_{N,t+1})$ and
  2. of non-investors valued at their gross return to k_{t+1} $(m_{N,t+1})$.

A DSGE model with Liquid and Illiquid Assets: Arbitrage

- The non-investor's Euler equation impose an arbitrage condition across the markets for capital and land.

- Arbitrage and competitive markets impose equality on the expected returns to capital and land weighted by the probability that next period the non-investing household may (or not) have the opportunity to invest

$$\Pi \mathbf{E}_t \left\{ \left[\frac{r_{L,t+1} + p_{t+1}}{p_t} - \frac{r_{K,t+1} + \lambda}{q_t} \right] \frac{1}{\mathcal{W}_{N,t+1|N,t} - \lambda(q_{t+1} - 1)k_{N,t+1}} \right\}$$

$$= \left(1 - \Pi\right) \mathbf{E}_t \left\{ \left[\frac{r_{K,t+1} + \lambda q_{t+1}}{q_t} - \frac{r_{L,t+1} + p_{t+1}}{p_t} \right] \frac{1}{\mathcal{W}_{N,t+1|N,t}} \right\}.$$

- Potential collateral constraints on future investors pushes p_{t+1} ($r_{L,t+1}$) above (below) that on capital $\Longrightarrow$ this is the liquidity premium on land driven by demand for its liquidity services.

A DSGE model with Liquid and Illiquid Assets: Investor Optimality I

- Solving an investing household's maximization problem yields an Euler equation that describes the investor's optimal decision in the capital market

$$\frac{1-\theta q_t}{(1-\theta)c_{I,t}} = \beta \mathbf{E}_t \left\{ \Pi \frac{r_{K,t+1}+\lambda}{c_{I,t+1|I,t}} + \left(1-\Pi\right)\frac{r_{K,t+1}+\lambda q_{t+1}}{c_{N,t+1|I,t}} \right\},$$

  where $c_{j,t+1|I,t}$ is consumption at $t+1$ of a household engaged in j, $j = I, N$, at that time, conditional on its investing at date t.

- The investor's capital Euler condition has an interpretation similar to that for the non-investor, except that
  1. a unit of date t consumption is valued at the investor's leverage
  2. and date $t+1$ marginal utility is conditioned on the household investing at date t.

A DSGE model with Liquid and Illiquid Assets: Investor Optimality II

- Similarly, the investing household has an Euler equation for its optimal decision making in the market for land.

- Since the investor is at a corner in its decision w/r/t holding land at date $t+1$, $m_{I,t+1} = 0$, the Euler equation for land of the investor does not hold with equality

$$\frac{p_t}{c_{I,t}} > \beta \mathbf{E}_t \left\{ \Pi \frac{r_{L,t+1} + p_{t+1}}{c_{I,t+1|I,t}} + \left(1 - \Pi\right) \frac{r_{L,t+1} + p_{t+1}}{c_{N,t+1|I,t}} \right\}.$$

- The cost of giving up a unit of consumption, valued in utils and priced at the cost of unit of land, is greater than the discounted expected benefit of holding land during date $t+1$.

A DSGE model with Liquid and Illiquid Assets: Summary

- Aggregation follows from the investor and non-investor optimality and equilibrium conditions because the probability a household becomes the former is *iid* across households and time.
- Given restrictions on preference and technology parameters, there is a steady state equilibrium in which
  1. Tobin's q holds, $q^* > 1 \Longrightarrow$ collateral constrained households,
  2. the liquidity premium drives $r_L^* < r_K^*$,
  3. the aggregate capital stock is lower than otherwise, and
  4. $m_I^* = 0 \Longrightarrow$ only non-investors own $\overline{L}$.
- The equilibrium is "Keynesian" in the sense that
  1. prices and returns, which are functions of quantities,
  2. generate fluctuations in quantities.
  3. $\Longrightarrow$ The equilibrium is not recursive.

Introduction: Jaccard (2013, or ECB WP1525, 2013)

- Jaccard argues that liquid assets are necessary for exchange of consumption goods.

- FIs produce liquid assets that they rent to firms and households.
  1. The stock of liquid assets is produced and owned by FIs.
  2. Liquid assets are in demand by households to buy the consumption good and by firms to pay for the capital they rent and wages to workers.
  3. Households and firms face liquidity risk $\Longrightarrow$ their demand for liquid assets may be greater than the supply.

- Claim that a liquidity crisis occurs when a small liquidity shock,
  1. that causes the supply of "safe" assets to shrink,
  2. increases transaction costs $\Longrightarrow$ reducing the volume of transactions and hence the level of real activity.

A DSGE Model with Liquidity in Advance Constraints

- The household and firm face liquidity in advance (LIA) constraints.
- One LAC limits date t consumption, c_t, by a fraction, θ, of the liquid asset, $s_{H,t}$, the household borrows from the FI during date t,

$$c_t \leq \theta s_{H,t}.$$

- The firm's total factor payments are bounded from above by a fraction, κ, of the liquid assets the firm rents from the FI,

$$w_t N_t + r_{K,t} K_t \leq \kappa S_{F,t},$$

where w_t, N_t, $r_{K,t}$, K_t, $S_{F,t}$ denote the real wage, labor demand, the real rental rate of capital, the firm's demand for capital, and the liquid assets the firm borrows from the FI during date t.

A DSGE Model with Liquidity in Advance Constraints

- Jaccard calls $\theta \in (0, 1)$ and $\kappa \in (0, 1)$ velocity parameters.
  1. θ restricts the liquidity preferences of the household,
  2. $\Longrightarrow s_{H,t}$ represents consumption possibilities for the household.
  3. κ affects the timing of factor demand by the firm
  4. $\Longrightarrow$ the firm's production possibilities are defined by $S_{F,t}$.
- Although $s_{H,t}$ and $S_{F,t}$ are choice variables for the household and firm, the aggregate stock of liquid assets, $s_{FI,t}$ is set by the FI $\Longrightarrow$ liquidity is endogenous.
  1. Intertemporal consumption and factor demand decisions respond to liquidity shocks.
  2. Liquidity shocks, which are driven by changes to θ and/or κ, differ from household liquidity preference and TFP shocks.

A DSGE Model with Liquidity in Advance Constraints: The Household

- The household wants to

$$\text{Max}_{\{c_t, n_t, i_t, k_{t+1}, h_{t+1}, s_{H,t}\}} \mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j \frac{\left[c_{t+j}\left[\psi + (1 - n_{t+j})^{\upsilon}\right] - h_{t-1+j}\right]^{1-\sigma}}{1-\sigma} \right\},$$

s.t. a budget constraint, the laws of motion of the stock of habits, h_{t+1}, and k_{t+1}, and the household's LIA constraint, $c_t \leq \theta s_{H,t}$, given k_0 and $h_0 > 0$.

- The budget constraint is $c_t + i_t + r_{S,t} s_{H,t} \leq d_{F,t} + d_{FI,t} + w_t n_t + r_{K,t} k_t$, where i_t, $r_{S,t}$, $d_{F,t}$ ($d_{FI,t}$), and n_t are investment, the rate a FI charges for liquidity, the dividends a final goods firm (FI) rebates to the household, and labor supply.

- The laws of motion of h_{t+1} and k_{t+1} are

$$h_{t+1} \leq \mu h_t + (1-\mu) c_{t+1}\left[\psi + (1 - n_{t+1})^{\upsilon}\right], \quad \mu \in (0, 1),$$

and

$$k_{t+1} \leq \left(1 + \left[\eta_1 + \frac{\eta_2}{1-\epsilon}\left(\frac{i_{t+1}}{k_t}\right)^{1-\epsilon}\right] - \delta\right) k_t, \quad \delta \in (0, 1).$$

A DSGE Model with Liquidity in Advance Constraints: The Firm

- The firm maximizes the expected discounted value of its profits,

$$\mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j \frac{\left[\psi + (1 - n_{t+j})^{\upsilon}\right] d_{F,t+j}}{\left[c_{t+j}\left[\psi + (1 - n_{t+j})^{\upsilon}\right] - h_{t-1+j}\right]^{\sigma}} \right\},$$

s.t. $d_{F,t} = y_t - w_t N_t - r_{K,t} K_t - r_{S,t} S_{F,t}$ and the firm's LIA constraint, $w_t N_t + r_{K,t} K_t \leq \kappa S_{F,t}$, where y_t is output of the economy's single consumption good.

- Since the household "owns" the firm, discount its future dividends at the time-varying rate of the marginal utility of consumption.

- The firm produces y_t using the CRS technology, $K_t^{\alpha}\left[A_t N_t\right]^{1-\alpha}$, where A_t is labor augmenting TFP and $\alpha \in (0, 1)$.

A DSGE Model with Liquidity in Advance Constraints: FIs

- A FI maximizes the expected discounted value of its profits,

$$\mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j \frac{\left[\psi + (1 - n_{t+j})^{\upsilon}\right] d_{FI,t+j}}{\left[c_{t+j}\left[\psi + (1 - n_{t+j})^{\upsilon}\right] - h_{t-1+j}\right]^{\sigma}} \right\},$$

s.t. $d_{FI,t} = r_{S,t}(s_{H,t} + S_{F,t}) - X_t$ and $S_{FI,t+1} = X_t + \varrho_t S_{FI,t}$, where X_t is the flow of new liquid assets into the stock of liquid assets the FI carries from date $t-1$ into date t and ϱ_t is a shock to the existing stock of liquid assets, given $S_{FI,0} > 0$.

- There is a balance sheet constraint, $s_{H,t} + S_{F,t} \leq \varrho_t S_{FI,t}$, facing the FI $\Longrightarrow$ the liquid assets lent to the household and the firm during date t is less than equal to the stock of liquid assets the FI owns gross of the liquidity shock ϱ_t.

- Since the household "owns" the FI, the discount on its dividends are time-varying using the marginal utility of household preferences.

A DSGE Model with Liquidity in Advance Constraints: Summary

- Liquidity shocks affect intertemporal consumption and capital accumulation decisions.
  1. Changes in the SDF and expected future returns to physical and "collateralized" assets drive current asset price movements.
  2. Preferences are nonseparable in consumption and leisure and intertemporally, which increase incentives to smooth utility.
  3. Claim is that these restrictions on utility generate positive comovement in c_t and n_t that are necessary to propagate a liquidity shock into a deep recession.
  4. $\Longrightarrow$ Asset prices fall in response to a liquidity shock.

- The FI accumulates the liquid asset for "free."
  1. Liquid assets need no factor inputs to produce, given the liquid asset initial condition, $S_{FI,0} > 0$, and the liquidity shock ϱ_t.
  2. Liquidity is not collateralized or backed by a real asset.

- If liquid assets are produced with real resources that are in limited supply, will asset prices fall in response to a liquidity shock?

A DSGE Model with Liquidity in Advance Constraints: Problem Set

- **I.** Construct and interpret the household's, firm's, and FI's FONCs.
- **II.** Construct and interpret the equilibrium and optimality conditions.
- Assume the TFP shock $\ln A_t$ is either,
  1. $\ln A_t = \nu + \gamma t + \rho_A \ln A_{t-1} + \xi_t$, $|\rho_A| < 1$ and $\xi_t \sim \mathcal{N}(0, \sigma_\xi^2)$,
  2. or $\ln A_t = \gamma + \ln A_{t-1} + \varphi_t$, $\varphi_t \sim \mathcal{N}(0, \sigma_\varphi^2)$.
  3. Assume the stationary component of the liquidity shock is $\hat{\varrho}_t = (1 - \rho_\varrho)\hat{\varrho}^* + \rho_\varrho \hat{\varrho}_{t-1} + \vartheta_t$, $|\rho_\varrho| < 1$, $0 \le \hat{\varrho}^*$, $\vartheta_t \sim \mathcal{N}(0, \sigma_\vartheta^2)$.
- **III.** Compute the steady state of the DSGE model with LACs when TFP (*i*) is a trend stationary AR(1) or (*ii*) is random walk with drift.
  1. What restrictions are necessary for the DSGE model to have a balanced growth path equilibrium under either TFP process? *Interpret*.
  2. Discuss how to calibrate DSGE model parameters to US data.
  3. Are there restrictions on preference and technology parameters necessary for c_t and n_t to have positive comovement? *Interpret*.
- **Extra Credit**: Add outside money, M_t, to the model in $c_t \le \theta s_{H,t} + M_t/P_t$ and to $c_t + i_t + r_{S,t}s_{H,t} + (M_{t+1} - M_t)/P_t \le d_{F,t} + d_{FI,t} + w_t n_t + r_{K,t}k_t$, where P_t is the aggregate price level. *Repeat* **I.**, **II.**, and **III.**
  Hint: You might need to include M_t and P_t elsewhere in the model.

Introduction

- This section extends the idea that small shocks in financial markets can produce large movements in asset prices and real activity.
- This is Allen and Gale's (Chapter 5) definition of *financial fragility*.
  1. Study a banking model with intrinsic and extrinsic uncertainty.
  2. What is the role of liquidity when market are incomplete?
  3. Facing collateral constraints, FIs sell assets to buy liquidity $\Longrightarrow$ generates volatile asset prices.
- Geanakoplos argues that collateral constraints changes the process that determines equilibria in financial markets.
  1. Expect collateral constraints to change in the future.
  2. Investors anticipate selling assets to satisfy debt obligations.
  3. $\Longrightarrow$ Leverage will fluctuate over the business cycle.
- Guerrieri and Lorenzoni (2011) explore the impact of a shock that dries up liquidity on households, given they are in debt and collateral constrained.

A Liquidity Preference Model of Banking

- Economy lasts three dates, $t = 0, 1, 2$.
  1. The single consumption good is t-dependent.
  2. Contracts struck at $t = 0$ while consumption at $t = 1, 2$.

- Households take addresses on the unit interval,
  1. are endowed with $x_0 = 1$ and $x_1 = x_2 = 0$,
  2. are ex ante identical, but ex post either consume at $t = 1$ with probability λ or at $t = 2$ with probability $1 - \lambda$.
  3. $\Longrightarrow$ Ex ante household preferences are $\lambda \mathcal{U}\left(c_1\right) + \left(1 - \lambda\right)\mathcal{U}\left(c_2\right)$.

- FIs offer households 1– and 2–period deposit contracts at $t = 0$.
  1. The former contract promises one unit of the good at $t = 1$.
  2. The latter contract promises $1 + R$ units of the good at $t = 2$.
  3. These returns are non-stochastic.
  4. During $t = 1$, the 1– and 2–period assets can be traded by the FIs in competitive markets.

A Liquidity Preference Model of Banking: Uncertainty

- Fundamental uncertainty is generated at the household, the FI, and aggregate levels.
  1. A household's liquidity preference is unknown ex ante.
  2. The fraction of early withdrawing depositors is unknown to FIs.
  3. The mass of early consuming households is unknown.
  4. Let the aggregate state of nature $s \in [1, 2]$ occur with probabilities $1 - \pi$ and π.
  5. Aggregate uncertainty dominates when the probability a household consumes early, $\lambda_1 < \lambda_2$, which is the probability of consuming late.

- Extrinsic uncertainty is the source of financial crises in the model given $\lambda_1 = \lambda_2$.

- Households learn their type at the beginning of $t = 1$, which is private information.

A Liquidity Preference Model of Banking: Asset Markets

- No markets exist for households to hedge their liquidity risk at $t = 0$ $\Longrightarrow$ asset markets are incomplete for households.
- At $t = 1$, asset markets are complete because (only) FIs can trade 1- and 2-period assets.
  1. Let p_s be the relative price of date $t = 2$ consumption per unit of date $t = 1$ consumption.
  2. $\Longrightarrow$ At $t = 1$, the price of the 2-period asset is $\mathcal{P}_s = p_s(1 + R)$.
  3. $\Longrightarrow$ FI assets are marked to market by competitive markets.

A Liquidity Preference Model of Banking: Financial Crises?

- Depositors run FIs when late consumers discover that $c_1 > c_2$ $\Longrightarrow$ the FI is known to be bankrupt.
- A rule of the game is that a FI has to satisfy every $t = 1$ consuming household that asks for its deposits.
- Assume $\mathcal{D}$ equal the promise a FI makes to $t = 1$ consumers.
- Let a FI invests a fraction $\mathcal{Y} \in (0, 1)$ in the 1-period asset at $t = 0$ $\Longrightarrow$ hold $1 - \mathcal{Y}$ in the 2-period asset.
  1. At $t = 1$, the value of the FI's assets are $\mathcal{Y} + (1 - \mathcal{Y})p_s(1 + R)$.
  2. At $t = 1$, the value of the FI's liabilities are $\lambda\mathcal{Y} + (1 - \lambda)p_s\mathcal{Y}$.
  3. $\Longrightarrow \lambda\mathcal{Y} + (1 - \lambda)p_s\mathcal{Y} \leq \mathcal{Y} + (1 - \mathcal{Y})p_s(1 + R)$ for the FI to pay off its depositors at $t = 1, 2$; otherwise depositors run the FI.
  4. $\Longrightarrow$ Bank runs are driven by expectations about the actual state of the FI's balance sheet.

A Liquidity Preference Model of Banking: The FI's Problem

- The FI maximizes $\mathbf{E}\Big\{\lambda_s \mathcal{U}\big(c_{1,s}\big) + \big(1-\lambda_s\big)\mathcal{U}\big(c_{2,s}\big)\,\Big|\, s\Big\}$ s.t. $0 \leq \mathcal{D}$ and $\mathcal{Y} \in (0,\,1)$, where $c_{1,s} = \mathcal{D}$ and $c_{2,s} = \dfrac{\mathcal{Y} + (1-\mathcal{Y})p_s(1+R) - \lambda_s\mathcal{D}}{(1-\lambda_s\big)p_s}$,
  1. if the incentive compatibility constraint (ICC) holds $\Longrightarrow \lambda\mathcal{Y} + (1-\lambda)p_s\mathcal{Y} \leq \mathcal{Y} + (1-\mathcal{Y})p_s(1+R)$;
  2. otherwise $c_{1,s} = c_{2,s} = \mathcal{Y} + (1-\mathcal{Y})p_s(1+R)$;
  3. $\Longrightarrow$ in either case the FI's budget constraint is satisfied.

- **Proposition 1**: Only if $p_s \leq 1$ does the asset market clear at $t = 1$ $\Longrightarrow$ when $p_s < (=)\ 1$ only the 2-period (1-period) asset is held.
  1. If $p_s > 1,\ P_s > 1 + R \Longrightarrow$ FIs never hold the 2-period asset at $t = 1$.
  2. The 2-period asset market does not exist at $t = 1$.
  3. $\Longrightarrow P_s = 1 + R$, FIs hold 1- and 2-period assets at $t = 1$ because both return one unit of consumption at $t = 2$ (no arbitrage).
  4. When $P_s < 1 + R$, $\mathcal{Y} = 0 \Longrightarrow$ at $t = 1$, the 2-period asset dominates the 1-period asset in rate of return.

A Liquidity Preference Model of Banking: Liquidity Shocks

- Suppose there is a small shock $\epsilon > 0$ to λ_S (this shock could be FI specific).
  1. A liquidity preference shock $\Longrightarrow \lambda_S + \epsilon$.
  2. A strictly positive ϵ can generate endogenous **intrinsic** bank runs.
  3. Unexpected demand for **liquidity** generates asset price volatility
  4. $\Longrightarrow$ FIs sell assets to satisfy early consumer demand for deposits.

- As $\epsilon \longrightarrow 0$, banks runs can still occur.
  1. FIs argee there is an **extrinsic** event that is tied to lower asset prices.
  2. When this event occurs, assets are sold by FIs at lower prices.
  3. Depositors observe the fall in asset prices $\Longrightarrow$ run because they anticipate FIs will violate their budget constraints and the ICC.

- This class of Allen and Gale bank run models differ from runs predicted by Diamond–Dybvig models.
  1. Diamond–Dybvig models are about a run on a single FI.
  2. Allen and Gale argue their bank run models characterize equilibria in which a crisis is **systemic** $\Longrightarrow$ asset price volatility affects all FIs.

Introduction

- Lenders often offer borrowers a debt contract that is tuple in a loan rate and a credit limit.
- This is odd . . . lenders set the price *and* quantity.
- Competitive markets clear by changing price and monopolists list prices and adjust their supply to satisfy demand.
- Geanakoplos argues that once default by borrowers is in question lenders respond by
    1. varying the loan to value (LTV) ratio with the state of the world.
    2. $\Longrightarrow$ Lenders impose collateral constraints on borrowers.
- If credit is a function of the state of the world, LTV ratios change either because asset prices move or collateral constraints do.

Disparities in the Demand for Assets

- What causes asset prices to fluctuate?
- Investors demand for assets differ.
  1. Preferences.
  2. Expectations of the future state of the world.
  3. Tulips $\Longrightarrow$ some are more beautiful.
- Changes in demand across asset classes and over time for assets generates asset price fluctuations.
- If collateral constraints adjust as asset prices change, leverage fluctuates $\Longrightarrow$ the leverage cycle.

The Leverage Cycle

- The leverage cycle reoccurs because
  1. lenders provide borrowers with too much credit $\Longrightarrow$ high leverage in a boom,
  2. a bad shock is realized (which could be extrinsic),
  3. asset prices fall, which
  4. forces creditors to tighten collateral constraints or to call loans.

- For Geanakoplos' model, debt needs to be collateralized by durable goods (or storable commodities).
  1. This is a *non-recourse* loan.
  2. The borrower secures the loan with something physical the lender can attach (*i.e.*, obtain) in case of a default.

- Credit market equilibrium is determined by the distribution of returns of different loans, which are functions of the (expected) payoffs on the physical stuff backing the debt.

The Leverage Cycle and Macroprudential Policy

- Is the leverage cycle something that should concern central banks?
- Should a central bank attempt to smooth the leverage cycle
    1. in the same way that interest rate rules are used
    2. in attempts to smooth shocks to the business cycle.
- Geanakoplos' model predicts that this "macroprudential" policy operates on the decisions of the marginal investor.
    1. The marginal investor determines the price of an asset $\Longrightarrow$ for example the winner's curse.
    2. Macroprudential policy functions to control the asset demand of just a few "outliers" in asset markets.

Introduction

- Guerrieri and Lorenzoni (GL) study a life-cycle model in which households are ex ante identical, but are ex post heterogeneous.

- Several important papers in this class of models are
  1. Bewley, T. (1977, "The Permanent Income Hypothesis: A Theoetical Formulation," *Journal of Economics Theory* 16, 252–292).
  2. Aiyagari, R. (1994, "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics* 109, 659-684).
  3. Huggett, M. (1993, "The Risk-Free Rate in Heterogeneous-Agent Incomplete- Insurance Economies," *Journal of Economic Dynamics and Control* 17, 953–69).
  4. Nakajima, M. (2012, "Business Cycles in the Equilibrium Model of Labor Market Search and Self-Insurance," *International Economic Review* 52, 399–432).
  5. Chapters 17 and 18 of Ljungqvist, L. and T.J. Sargent (2012, Recursive Macroeconomic Theory, third edition, Cambridge, MA: MIT Press).

- These models are solved using numerical methods.

The GL Aiyagari–Huggett Model: Brief Review

- Since households plan over a life-cycle, they want to smooth consumption as in any PIH model.

- Households can be creditors or debtors as they smooth consumption, given
  1. labor supply decisions face idiosyncratic transitory income shocks,
  2. there is an exogenous borrowing limit, no physical capital, and the only asset is a unit discount bond $\Longrightarrow$ incomplete financial markets.

- GL explore the response of households to an *aggregate* shock that tightens collateral constraints for all households.
  1. Debtors react to tighter collateral constraints by paying off their loans and not taking on new loans.
  2. $\Longrightarrow$ Deleveraging by debtors as they self-insure against future (bad) idiosyncratic income shocks.
  3. Creditors also self-insure by increasing their savings.
  4. $\Longrightarrow$ Precautionary motive driven by third derivative of a household's utility function.

The GL Aiyagari–Huggett Model: Households

- A household, taking address i on the unit interval, has lifetime preferences

$$\mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \beta \mathcal{U}\left(c_{i,t}, n_{i,t}\right) \middle| \theta_{i,t} \in \Theta_S \right\}, \ \ \beta \in (0, 1),$$

where $c_{i,t}$, $n_{i,t}$, and $\theta_{i,t}$ are its consumption, labor supply, and its idiosyncratic labor supply shock, which follows a first-order Markov chain drawn from $\Theta_S = [\theta_1 \ \ldots \ \theta_s]'$.

- The technology, $\theta_{i,t} n_{i,t}$, generates output, $y_{i,t}$, for the ith household.
  1. There are no aggregate shocks in GL's baseline model.
  2. Normalize $\theta_1 = 0 \Longrightarrow$ household is in the unemployment state.

- The household's budget constraint is $q_t b_{i,t+1} + c_{i,t} + \tilde{\tau}_{i,t} \leq y_{i,t} + b_{i,t} + \upsilon_{i,t}$, where q_t is the price of the unit discount bond $b_{i,t+1}$ household i carries into t+1 from t, $\tilde{\tau}_{i,t}$ is tax household i pays that is a common lump sum tax, τ_t, net of household i's transfer $\upsilon_{i,t}$ from the government $\Longrightarrow$ when $\theta_{i,t} = \theta_1 = 0$, $\upsilon_{i,t} > 0$ and $\tilde{\tau}_{i,t} = \tau_t - \upsilon_{i,t}$; otherwise $\tilde{\tau}_{i,t} = \tau_t$ and $\upsilon_{i,t} = 0$.

- All households face the exogenous borrowing constraint $\phi > 0 \Longrightarrow b_{i,t+1} \geq -\phi$.
  1. Common *ad hoc* bound on debt accumulation, $b_{i,t+1} < 0$, by households.
  2. The *natural debt* limit of household i is the expected PDV of its labor income $\Longrightarrow b_{i,t+1} \geq -\mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \left[\Pi_{\ell=0}^{j} (1 + r_{t+\ell}) \right]^{-1} \theta_{i,t+j} \, n_{i,t+j} \right\}$, where $1 + r_t = q_t^{-1}$.
  3. GL want the natural debt limit smaller than $-\phi$.
  4. The necessary assumption is $\upsilon_{i,t} > 0$.

The GL Aiyagari–Huggett Model: Equilibrium and Optimality

- The date t state vector is $(b_i, \theta_i) \Longrightarrow c_{i,t} = C_t(b_i, \theta_i)$ and $n_{i,t} = \mathcal{N}_t(b_i, \theta_i)$.
- Substitute $C_t(b_i, \theta_i)$ and $\mathcal{N}_t(b_i, \theta_i)$ into the household budget constraint
  1. to obtain $b_{i,t+1}$ given r_t, b_t, and τ_t for $t = 1, 2, \ldots \infty$.
  2. $\Longrightarrow$ a density function $\Psi_t(b_i, \theta_i)$ that maps the state from t into $t+1$.
- **Definition 1**: An equilibrium is a sequence of endogenous relative prices $\{r_t\}_{t=1}^{\infty}$ and exogenous fiscal shocks $\{\tau_t\}_{t=1}^{\infty}$, a sequence of decisions $\left\{C_t(b_i, \theta_i), \mathcal{N}_t(b_i, \theta_i)\right\}_{t=1}^{\infty}$ and a sequence of state transition distributions $\Psi_t(b_i, \theta_i)$, given $\Psi_0(b_i, \theta_i)$, such that
  1. $\left\{C_t(b_i, \theta_i), \mathcal{N}_t(b_i, \theta_i)\right\}_{t=1}^{\infty}$ are optimal given $\{r_t\}_{t=1}^{\infty}$ and $\{\tau_t\}_{t=1}^{\infty}$,
  2. the law of motion $\Psi_t(b_i, \theta_i)$ conforms with 1.,
  3. the government budget constraint holds, $\tau_t = B_t - r_t B_{t+1}/(1 + r_t) + u\upsilon_t$, where $\overline{B} = B_t$ for all t and $u = Pr(\theta_i = \theta_1 = 0)$, and
  4. the bond market clears, $\overline{B} = \int b_{i,t}\Psi_t(b_i, \theta_i)di$, otherwise $r_t \longrightarrow \infty$.
- Along any equilibrium path, optimality requires $\theta_{i,t}\mathcal{U}_1\left(c_{i,t}, n_{i,t}\right) = \mathcal{U}_2\left(c_{i,t}, n_{i,t}\right)$ given $\theta_{i,t} \neq \theta_1$, and $\mathcal{U}_1\left(c_{i,t}, n_{i,t}\right) \geq \beta(1 + r_t)\mathbf{E}_t\left\{\mathcal{U}_1\left(c_{i,t+1}, n_{i,t+1}\right)\right\}$.
  1. *Intratemporal* optimality is satisfied trivially when $\theta_{i,t} = \theta_1 \Longrightarrow n_{i,t} = 0$.
  2. *Intertemporal* optimality holds with equality if $b_{i,t+1} > -\phi$.

The GL Aiyagari–Huggett Model: Summarize Results

- Relative prices and real activity responds to an aggregate shock to ϕ by following $\Psi_t(b_i, \theta_i)$ to a new steady state.
- The most indebted households aim to smooth consumption
  1. by deleveraging quickly given the shock to $\phi \Longrightarrow$ consumption smoothing reinforced by precautionary motive,
  2. lower consumption and/or supply more hours to the market to raise their labor income.
  3. Given r_t, the drop in $c_{i,t}$ appears to be a preference shock to $\beta \Longrightarrow$ an increase in $\mathcal{U}_1\left(c_{i,t}, n_{i,t}\right)$ produces a "Jensen effect" in $\mathbf{E}_t\left\{\mathcal{U}_1\left(c_{i,t+1}, n_{i,t+1}\right)\right\}$.
- Deleveraging generates two different interest rate responses.
  1. The borrowing/lending rate jumps down at impact.
  2. At longer horizons, the interest rate rises to a new steady state as households adjust their portfolios in response to an increase in $\phi \Longrightarrow$ tighter lending standards generates overshooting or a humped shape response in r_t.
- Whether a indebted household lowers consumption less/more than its labor supply depends on the respective interest rate and income elasticities.
- Results are robust to adding nominal frictions and durable goods.

Introduction

- Holmström and Tirole tie risk premium to the lack of pledgeable income.
  1. At the start of a project, an entrepreneur cannot credibly pledge the project's future income to investors.
  2. Risk premium ⇔ liquidity premium, but only on short term assets.

- Jermann and Quadrini (JME, 2007) address the question of the sources and causes of the tech boom and bust in the U.S. of late 1990s.
  1. Construct a RBC model in which entrepreneurs have access to production technologies exhibiting diminishing returns to scale, but
  2. there is the potential entrepreneurs will abscond with some of the proceeds of a project promised to investors ⟹ some project income is not pledgeable.
  3. Liquidity shortage ⟹ initial project size smaller than optimal,
  4. the size distribution of firms is endogenous and evolves over time, and
  5. aggregate productivity is endogenous because it is a function of the size distribution of firms.

- Bigio (2012) & Shi (JME, 2015) study liquidity/collateral constraints in RBC models.
  1. The deep structure of liquidity and collateral constraints may be observationally equivalent to risk premium and/or net worth shocks.
  2. However, these shocks need extraordinary persistence and volatility to replicate observed fluctuations in asset prices/returns and real activity.
  3. Liquidity and collateral constraints are missing something important going on financial markets.

Holmström and Tirole: Liquidity Asset Pricing, I

- Suppose entrepreneurs are heterogeneous, but investors are not.
  1. Some firms have excess liquidity in good states of the world while the liquidity demand of other firms is not state contingent.
  2. Investors are risk neutral $\Longrightarrow$ consumption is not state contingent $\Longrightarrow$ invest at $t=0$ and expect to "consume" the same bundle in every state at $t=1$.

- Assets are priced at $t=0 \Longrightarrow$ the $t=0$ price of liquidity at $t=1$.
  1. The two states are a high, H, liquidity shock state and low, L, state.
  2. The probability of the H (L) liquidity shock state is f_H (f_L), where $f_H + f_L = 1$.
  3. The prices of liquidity in the H and L shock states are s_H and s_L, where $s_j \geq 1$, $j = H, L$, because investors' consumption is not state contingent.
  4. $\Longrightarrow$ The state contingent price of liquidity in state $j = s_j f_j$.

Holmström and Tirole: Liquidity Asset Pricing, II

- What is the non-state contingent or unconditional price of liquidity?
  1. Suppose there is excess demand for liquidity in state $H \Longrightarrow s_H > 1$, but not in state $L \Longrightarrow s_L = 1$.
  2. The price of liquidity $q = f_H s_H + f_L \times 1 \Longrightarrow$ the liquidity premium is $q - 1 = f_H(s_H - 1)$.

- A risk free asset provides non-state contingent liquidity in every state
  1. The yield, κ, on this asset is $\dfrac{1-q}{q} = -\dfrac{f_H(s_H - 1)}{1 + f_H(s_H - 1)} \leq 0$.
  2. If $q = 1$, the liquidity premium $= 0$, $\kappa = 0$, and $s_H = 1 \Longrightarrow$ not excess demand for liquidity in state H.
  3. Otherwise, $\kappa < 0 \Longrightarrow$ suppliers of liquidity charge a premium to buyers, but this is not a discount on future consumption.

- Suppose entrepreneurs issue long-term bonds with a par value $= 1$ and a maturity of n periods to fund projects.
  1. Assume investors expect payoffs of $\theta_{L,n} = 1$ and $\theta_{H,n} < 1 \Longrightarrow$ partial default in the H liquidity shock state.
  2. $\Longrightarrow$ At t=0, the price of the bond $q_n = f_H s_H \theta_{H,n} + f_L \times 1$ and
  3. its yield is $\kappa_n = \dfrac{f_H \theta_{H,n} - q_n}{q_n} = \dfrac{f_H \theta_{H,n}(1 - s_H)}{1 - f_H(1 - \theta_{H,n} s_H)} \Longrightarrow \kappa_n - \kappa > 0$.
  4. The term premium, $\kappa_n - \kappa$, is positive because there is a potential for the risky bond to default (partially), $\theta_{H,n} < 1 \Longrightarrow$ in HT's model, the term premium on risky assets should always dominate the liquidity premium.

Introduction: Jermann and Quadrini (JME, 2007)

- JQ construct a RBC model in which expectations of shifting between a low growth and a "new economy" generate booms and busts.
  1. Expectations are grounded in fundamentals of TFP growth and a first-order Markov chain that sets the TFP growth regime $\Longrightarrow$ there are two TFP regimes.
  2. One TFP regime is low growth. The other is high growth $\Longrightarrow$ the new economy.
- The endogenous mechanisms of the RBC model are
  1. entrepreneurs do not necessarily obey debt contracts $\Longrightarrow$ limited enforcement of collateral constraints and
  2. diminishing marginal returns (DMR) at the firm level.
- Entrepreneurial projects or firms are heterogeneous w/r/t to size.
  1. Firm size is a function of age $\Longrightarrow$ as firms age they grow or cease to exist.
  2. $\Longrightarrow$ The size distribution of firms is endogenous and dynamic.
  3. TFP shocks are common across firms, but aggregate productivity is a function of the size distribution of firms.
- Ex: Switch to the "new economy' regime and raise expectations of future growth.
  1. The expected PDV of firm profits rise $\Longrightarrow$ eases collateral constraints giving a boost to the size and marginal products of new projects.
  2. Existing firms respond by shrinking to equate MPs across firms.
  3. $\Longrightarrow$ Reallocates factor inputs from older to younger firms.
  4. The distribution of firm size becomes more peaked, which raises aggregate productivity $\Longrightarrow$ DMR

Jermann and Quadrini: Households and Their Preferences

- The unit mass of households have finite lives.
  1. Let α = probability of surviving from date t to date $t+1$ $\Longrightarrow$ the mass of new households is $1 - \alpha$.
  2. A fraction e of new households are endowed with an investment project $\Longrightarrow$ they become entrepreneurs given financing.
  3. The remaining new households, $1 - e$, become workers.

- Household preferences are $\mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \left(\frac{\alpha}{1+r} \right)^j \left[c_t - \varphi_t(h_t) \right] \right\}$, where
  1. r, c_t, h_t, and $\varphi_t(\cdot)$ are the household's subjective rate of time preference, consumption, hours worked, and the disutility of work.
  2. Preferences are risk neutral w/r/t $c_t \Longrightarrow r$ is the riskless rate.
  3. JQ need a balanced growth path equilibrium $\Longrightarrow$ a time-varying disutility of work, where $\varphi_t(0) = 0$ (for entrepreneurs, $h_t = 0$) and $\varphi_t(\cdot)$ is strictly convex for all dates $t \Longrightarrow$ its a cost function.

- A worker's labor market optimality condition is $\varphi_t'(h_t) = w_t/(1+r)$, where workers are paid tomorrow for supply labor today, w_t is the (real) wage, and $\varphi_t'(h_t)$ implies a labor supply function $\Longrightarrow h_t = \varphi_t'^{-1}\left(\frac{w_t}{1+r} \right)$.

Jermann and Quadrini: Technology, Project Finance, and Entrepreneurs

- Entrepreneurs need capital, k_t, and labor, ℓ_t, to run
  1. the CRS technology $\mathcal{F}(k_t, \ell_t)$, which uses date t inputs to produce date $t+1$ output, $y_{t+1} = z_t \mathcal{F}(k_t, \ell_t)^\theta$,
  2. where z_t is TFP common to all firms $\Longrightarrow$ aggregate productivity, and
  3. $\theta \in (0, 1) \Longrightarrow y_{t+1}$ is produced with decreasing returns to scale (DRS).
  4. Assume a fixed start up cost κ_t to produce y_{t+1}.

- Projects cease operation either because
  1. an entrepreneurs dies (with probability $1 - \alpha$) or the project is no longer productive with probability $1 - \phi_t$.
  2. $\Longrightarrow$ An entrepreneur's project survives with probability $\alpha\phi_t$,
  3. where ϕ_t is generated by a first-order Markov chain, which is rigged to deliver a falling survival probability as firm ages.

- New projects need more k_t than entrepreneurs own.
  1. Entrepreneurs borrow from investors, but complete enforecement of debt contracts is not possible $\Longrightarrow$ similar to HT's problem of an entrepreneur not being able to credibly pledge all the proceeds of an investment project.
  2. An entrepreneur can draw a private return or income from the project, $\mathcal{D}_t = \lambda y_{t+1}$, $\lambda > 0$, $\Longrightarrow \mathcal{D}_t$ is the value of the firm at default.
  3. If $\mathcal{D}_t > 0$, the firm stops production and its k_t has zero value in outside alternatives $\Longrightarrow$ full depreciation or $\delta = 1$.
  4. $\Longrightarrow 1 - \lambda =$ cost to investor of default by a firm per unit of $k_t \Longrightarrow$ costly state verification problem of Williamson (1987).

Jermann and Quadrini: Aggregate TFP Shocks and Balanced Growth

- TFP growth: $g_{z,t} = (g_{z,L}, g_{z,H})$, $0 < g_{z,L} < g_{z,H}$, and $g_{z,t} > 0$ for all t.
- There are $i = 1{,}2$ first-order processes generating serial correlation in $g_{z,t}$.
  1. $\Longrightarrow$ TFP is subject to regime shifts, where regime shift process is the first-order Markov chain $\Upsilon(i' \mid i)$, where i' is the date $t+1$ state, $i = 1, 2$.
  2. Denote the transition probability $\Gamma_i(g_z' \mid g_z)$, $i = 1, 2$, where $g_z' \equiv g_{z,t+1}$.
- Strictly positive $g_{z,t} \Longrightarrow$ no limit on $z_t \Longrightarrow$ it grows without bound.
  1. The equilibrium path of the economy has to be balanced $\Longrightarrow$ otherwise equilibrium decision rules are not convex.
  2. Let $1 + g_{z,t} = (1 + g_t)^{1-\theta\epsilon}$ and $\mathcal{A}_t = \prod_{j=0}^{t-1}(1 + g_{t-j})$ and specify $\mathcal{F}(k_t, \ell_t) = k_t^{\epsilon}\,\ell_t^{1-\epsilon}$, $\epsilon \in (0, 1)$, $\varphi_t = \chi\mathcal{A}_t h^{\nu}$, and $\kappa_t = \kappa\mathcal{A}_t$, $\nu, \kappa > 0$,
  3. to insure convex decision rules that generating equilibrium outcomes fluctuating around the stochastic trend $\mathcal{A}_t$.
  4. $\Longrightarrow \ln\mathcal{A}_t \approx \ln z_t - \ln z_1 \Longrightarrow$ divide by t = annual average stochastic growth rate, which is used for stochastic detrending.

Jermann and Quadrini: Asset Pricing

- Firms leave date t with revenue equal to $\mathcal{R}_t = (1-\delta)k_t + z_t\mathcal{F}(k_t, \ell_t)^{\theta} - w_t\ell_t$.
- FONC w/r/t ℓ_t: $w_t = \theta z_t\mathcal{F}(k_t, \ell_t)^{\theta-1}\mathcal{F}_{\ell}(k_t, \ell_t) = \theta z_t\mathcal{F}_{\ell}(k_t, \ell_t)^{\theta} \Longrightarrow \mathcal{F}(\cdot, \cdot)$ is CRS.
  1. $\Longrightarrow$ The labor demand schedule $\ell_t = \theta^{-1}z_t^{-1}\mathcal{F}_{\ell}^{-1}(k_t, w_t)^{\theta}$ and $k_t/\ell_t = f(w_t)$.
  2. $\Longrightarrow$ A firm's revenue is a function of TFP, capital and the real wage

$$\mathcal{R}(z_t, k_t, w_t) = (1-\delta)k_t + z_t\mathcal{F}(k_t, \ell(z_t, k_t, w_t))^{\theta} - w_t\ell(z_t, k_t, w_t).$$

- The market value of a firm at date t is a function of the future expected stream of $\mathcal{R}(z_t, k_t, w_t)$ and whether the firm will continue into date $t+1$.
  1. If the firm survives to date $t+1 \Longrightarrow \mathcal{R}(z_t, k_t, w_t) - k_{t+1}$, and otherwise
  2. $k_{t+1} = 0$, the firm defaults at the end of date t leaving $\mathcal{R}(z_t, k_t, w_t)$.
- Suppose there exists an equity market in which claims on "dividends" are traded.
  1. Dividends are $\mathcal{R}(z_t, k_t, w_t) - k_{t+1}$ for ongoing firms $\Longrightarrow$

$$\mathcal{P}_t = \left(\frac{1}{1+r}\right)\mathbf{E}_t\left\{\sum_{j=0}^{\infty}\left(\prod_{s=j}^{t-1}\beta_s\right)\left[\mathcal{R}(z_{t+j}, k_{t+j}, w_{t+j}) - \alpha\phi_{t+j}k_{t+1+j}\right]\right\}.$$

  where $\beta_s = \alpha\phi_t/(1+r)$ and $\alpha\phi_t$ is the probability of a firm's survival.
  2. Stochastic detrending yields

$$\mathcal{P}_t = k_t - \mathbf{E}_t\left\{\sum_{j=0}^{\infty}\left[\prod_{s=j}^{t-1}\beta_s(1+g_{s+1})\right]\left[\left(\frac{1}{1+r}\right)\mathcal{R}(k_{t+j}, w_{t+j}) - k_{t+j}\right]\right\}.$$

Jermann and Quadrini: Timing of Events and Financial Contracts

- Workers and entrepreneurs learn about ϕ_t, z_t, $\Gamma_i(g'_z \,|\, g_z)$, and $\Upsilon(i' \,|\, i)$ at the beginning of date t.

- A firm owns $(1-\delta)k_{t-1} + z_{t-1}\mathcal{F}(k_{t-1}, \ell_{t-1},)^{\theta}$, which are used to pay
  1. the wage bill $w_t\ell_t$ and investing in new capital, k_t, which implies dividends.
  2. Given these actions, the firm hires ℓ_t to produce y_{t+1} and decides whether or not to default on its debt.
  3. $\Longrightarrow$ Entrepreneur reneges on debt prior to observing z_{t+1}, which is key to designing the debt contract.

- Assume debt contracts are always obeyed by entrepreneurs.
  1. $\Longrightarrow$ Equilibrium has ex post identical entrepreneurs w/r/t their investment decision, $k_t = \overline{k}$ for all dates t, where
  2. $\overline{k} = \arg\max_k \left\{\left(\dfrac{1}{1+r}\right)\mathcal{R}(k, w) - k\right\}$ $\Longrightarrow$ the mass of firms is constant at every moment in time without default.
  3. $\Longrightarrow$ Start up costs and labor supply are constant along the balanced growth path $\Longrightarrow$ $\hat{\kappa}_t = \kappa$ and $\varphi(h) = \chi h^{\nu}$, but $\ell_t = \theta^{-1}z_t^{-1}\mathcal{F}_{\ell}^{-1}(\overline{k}, w)^{\theta}$.

- Suppose this economy is subject to regime switch from i to i'.
  1. $\Longrightarrow$ Shift in the Markov chain generating g_t, which alters the expected discounted value of firms.
  2. $\Longrightarrow$ Change (conditional) probability density function of TFP growth drives change (conditional) probability density function of $\mathcal{P}_t$.
  3. However, there is no impact on current decisions only on future decisions.

Jermann and Quadrini: Financial Contracts with Default

- Suppose the economy also has investors offering entrepreneurs a financial contract that conditions on the potential for default.

- The contract imposes four constraints on entrepreneurs.
  1. The contract promises an entrepreneur an optimal stream of consumption $\Longrightarrow q \geq \beta \mathbf{E}\Big\{(1+g')[c(\mathbf{S}') + q(\mathbf{S}')]\Big\}$, where q is the value of the contract to the entrepreneur, $\mathbf{S}' = (\mathbf{s}', \phi')$, and $\mathbf{s}'$ is a vector of z', $\Gamma_i(g_z' \,|\, g_z)$, and $\Upsilon(i' \,|\, i)$.
  2. The ICC is $\mathcal{D}\Big(k,\, w(\mathbf{S}')\Big) \leq \beta \mathbf{E}\Big\{(1+g')[c(\mathbf{S}') + q(\mathbf{S}')]\Big\}$, from which the optimal k is set by the financial contract.
  3. There are two non-negativity constraints, $c(\mathbf{S}') \geq 0$ and $q(\mathbf{S}') \geq 0 \Longrightarrow$ investors promise neither to borrow nor impose transfers on entrepreneurs.
  4. The remainder of the value of the contract is claimed by the investor.

- Let $\mathcal{V}(\mathbf{S}', q)$ = aggregate value of the contract at end of the current period net of k $\Longrightarrow$ investor solves the dynamic program to construct the optimal contract under default

$$\mathcal{V}(\mathbf{S}', q) = \text{Max}_{k,\, c(\mathbf{S}'),\, q(\mathbf{S}')} \left[\left(\frac{1}{1+r} \right) \mathcal{R}\Big(k, w(\mathbf{S})\Big) - k + \beta \mathbf{E}\Big\{(1+g')\mathcal{V}\Big(\mathbf{S}',\, q(\mathbf{S}')\Big)\Big\} \right],$$

s.t., the promise, IC, and non-negativity constraints.

Jermann and Quadrini: Optimality Conditions of the Financial Contract

- The FONCs w/r/t k, $c(\mathbf{S}')$, and $q(\mathbf{S}')$ are
  1. $1 = \left(\frac{1}{1+r}\right)\mathcal{R}_k\Big(k,\ w(\mathbf{S})\Big) - \lambda_2\mathcal{D}_k\Big(k,\ w(\mathbf{S})\Big)$,
  2. $\lambda_1 - \lambda_2 = 0$, given an interior solution $c(\mathbf{S}') > 0$,
  3. $\lambda_1(\mathbf{S}') - \lambda_1 + \lambda_2 = 0$, where $\lambda_1(\mathbf{S}') = \beta\mathbf{E}\Big\{(1+g')\mathcal{V}_q\Big(\mathbf{S}',\ q(\mathbf{S}')\Big)\Big\}$ for all $\mathbf{S}'$,
  4. $q = \beta\mathbf{E}\Big\{(1+g')[c(\mathbf{S}') + q(\mathbf{S}')]\Big\}$, and
  5. $q - \mathcal{D}\Big(k,\ w(\mathbf{S})\Big) \geq 0$, given $\lambda_2 > 0$, but $c(\mathbf{S}') = 0$, where
  6. λ_1 and λ_2 are the Lagrange multipliers tied to the promise and IC constraints.

- W/r/t k: Cost of investing one unit of k = discounted value of the marginal revenue contributed by the one unit of k net of the marginal change in the default value of the investment priced at the marginal value of the entrepreneur's share of the contract.

- W/r/t $c(\mathbf{S}')$ and $q(\mathbf{S}')$: $c(\mathbf{S}') = 0$ if $\lambda_1(\mathbf{S}') > 0 \Longrightarrow \lambda_1$ falls when the ICC binds, but as $\lambda_1(\mathbf{S}') \longrightarrow 0$, expectations are the ICC will not bind in the future $\Longrightarrow \lambda_2 = 0$ for all $\mathbf{S}'$.
  1. When entrepreneurs expect not to be constrained in the future, $\lambda_1(\mathbf{S}') = 0$ $\Longrightarrow$ the FONC w/r/t k yields the optimal $\overline{k}(\mathbf{S}')$.
  2. However, the choice of k and ℓ is suboptimal for the entrepreneur along the path of $\lambda_1(\mathbf{S}') \longrightarrow 0$, because $\lambda_2 > 0$ until $\lambda_1(\mathbf{S}')$ reaches zero.
  3. While the ICC still binds, $q = \mathcal{D}\Big(k,\ w(\mathbf{S})\Big)$ and $c(\mathbf{S}') = 0 \Longrightarrow$ expectations of future TFP drive decisions about k_t and ℓ_t because in the current state of the economy the promise and IC constraints bind.

Jermann and Quadrini: Equilibrium

- **Proposition 1**: There exists $\overline{q}(\mathbf{S}')$ such that
  1. The value function $\mathcal{V}(\mathbf{S}', q)$ is increasing and concave in $q \geq \overline{q}(\mathbf{S}')$.
  2. The choice of k is the minimum value on the real line between $k = \mathcal{D}^{-1}\Big(q, w(\mathbf{S})\Big)$ and $\overline{k}(\mathbf{S}')$.
  3. If $q \leq \beta\mathbf{E}\Big\{(1+g')q(\mathbf{S}')\Big\}$, the solution $c(\mathbf{S}') = 0$ is unique.
  4. If $q > \beta\mathbf{E}\Big\{(1+g')q(\mathbf{S}')\Big\}$, $c(\mathbf{S}') > 0$ with a multiplicity of solutions.

- **Proof**: Rely on the concavity of the debt contract's dynamic program.
  1. The trick is to hold the DRS "fixed" $\Longrightarrow \mathcal{V}(\mathbf{S}', q)$ is unique.
  2. The dynamic program satisfies the conditions for Bellman's equation $\Longrightarrow$ necessary conditions for optimality and a contraction mapping.

- Along the equilibrium path, an entrepreneur chooses k, $c(\mathbf{S}')$, and $q(\mathbf{S}')$ to fluctuate around the balanced growth path.
  1. This process stops when $q(\mathbf{S}') = \overline{q}(\mathbf{S}')$ and $k = \overline{k}(\mathbf{S}')$.
  2. Given the entrepreneur invests $\overline{k}(\mathbf{S}) \Longrightarrow \mathcal{P}(\mathbf{S}) = \overline{k}(\mathbf{s}) + \mathcal{V}\Big(\mathbf{S}, \overline{q}(\mathbf{S})\Big)$.

Jermann and Quadrini: The Initial Financial Contract

- Investors offer entrepreneurs debt contracts in a competitive debt market.
- When a entrepreneur enters the debt market the first time
  1. $\mathcal{V}(\mathbf{S}', q) \geq q + \kappa \Longrightarrow$ the value of the contract equals at least the entrepreneur's valuation of the contract plus the stochastically detrended fixed cost of starting the investment project or zero profits for investors.
  2. Investors offer $q_0(\mathbf{S}') = \text{Max } q$, s.t. the zero profit condition.
  3. Since Proposition 1 establishes $\mathcal{V}(\mathbf{S}', q)$ is increasing, concave, and has zero slope when $q \geq \overline{q}(\mathbf{S}')$.
  4. $\Longrightarrow$ The investors' zero profit condition binds.
  5. $\Longrightarrow \mathcal{V}(\mathbf{S}', q) - q$ falls as q rises if $q \geq \overline{q}(\mathbf{S}')$.
  6. As q rises, k is higher $\Longrightarrow$ a larger initial investment and labor demand, which increases aggregate capital and employment.

Jermann and Quadrini: The Recursive Equilibrium

- **Definition**: A recursive equilibrium consists of
  1. the consumption function $c(\mathbf{S})$ and labor supply function $h(\mathbf{S})$,
  2. the financial contract's value function $\mathcal{V}(\mathbf{S}, q)$,
  3. the investment function $k(\mathbf{S}, q)$, consumption function $c(\mathbf{S}, q)(\mathbf{S}')$, and the value of debt contract $q(\mathbf{S}, q)(\mathbf{S}')$ for the entrepreneur,
  4. the initial value of the financial contract for new entrepreneurs $q_0(\mathbf{S})$,
  5. the wage function $w(\mathbf{S})$, which workers and firms take parametrically,
  6. aggregate investment equals aggregate savings from workers and entrepreneurs, labor supply equals labor demand, and
  7. the law of motion of the exogenous state vector $\mathbf{S}'$.
  8. $\Longrightarrow$ Worker and entrepreneurial decisions are optimal,
  9. $w(\mathbf{S})$ clears the labor market, and the goods market clears, and
  10. workers and entrepreneurs hold rational expectations $\Longrightarrow$ their subjective beliefs about the law of motion of $\mathbf{S}'$, $q_0(\mathbf{S})$, $q(\mathbf{S})$, and $w(\mathbf{S})$ match market outcomes.

Jermann and Quadrini: Summary

- JQ show that expectations about future TFP affect financial constraints, which alter current production decisions $\Longrightarrow$ a KM-like result.
- Layer on top of this mechanism regime shifts in the mean TFP growth rate that generate boom and busts in "equity" prices.
- This RBC model also generates a positive correlation between current labor input and future productivity.
- This is not the same as having a DSGE model that predicts movements in asset prices are a source of business cycle fluctuations.
- Still, JQ's RBC model suggests a mechanism to transmit financial market shocks to the real economy.

Introduction: Bigio (Liquidity Shocks and the Business Cycle: What next?)

- NK-DSGE model often include ad hoc/reduced form shocks interpreted as financial market disturbances.
  1. Preference shock to real sovereign debt in the utility function $\Longrightarrow$ a risk premium shock.
  2. A shock to investment adjustment costs $\Longrightarrow$ a shock to net worth.
- These are "observational equivalence" results.
  1. A preference shock to the utility value of the transactions services provided by real sovereign debt in that along the equilibrium path this economic primitive acts as if its a risk premium shock.
  2. Shock to the cost of adjusting investment alters the market value of capital to its replacement cost, which is a shock the BGG external finance premium $\Longrightarrow$ a shock to the spread on the return to projects over the riskless rate.
- Observational equivalence suggests these shocks and the associated utility and technology functions are not deep economic structure.

A DSGE Model with Wealth in Utility and a Preference Shock

- Suppose period utility of the representative household is $\ln c_t + \alpha\mathcal{V}(1 - n_t) + \varrho_t\mathcal{W}(B_{t+1}/P_t)$,
  1. where c_t, n_t, ϱ_t, B_{t+1}, and P_t are consumption, labor supply, a preference shock to the household's utility value of real government debt, one-period government debt, and the aggregate price level and the household discount factor is $\beta \in (0, 1)$.
  2. The budget constraint of the household is

$$c_t + x_t + B_{t+1}/P_t \leq r_t k_t + w_t n_t + R_{B,t-1}B_t/P_t,$$

     where x_t, r_t, w_t, and $R_{B,t-1}$ are investment, the real rental rate of capital, k_t, the real wage, and the nominal return on B_t.
  3. The law of motion of k_{t+1} is

$$k_{t+1} = (1 - \delta)k_t + \left[1 - \mathcal{Q}\left(x_t/x_{t-1}\right)\right]x_t,$$

     where $\mathcal{Q}(x_t/x_{t-1})$ is a cost of adjustment function in investment growth, x_t/x_{t-1}.

Identifying a Risk Premium Shock

- The FONC of B_{t+1} is $\lambda_{1,t} - \varrho_t \mathcal{W}'(b_{t+1}) = \beta R_{B,t} \mathbf{E}_t \left\{ \lambda_{1,t+1}/\pi_{t+1} \right\}$, where $\lambda_{1,t}$ is the Lagrange multiplier on the budget constraint, $b_{t+1} = B_{t+1}/P_t$, and $\pi_{t+1} = P_{t+1}/P_t$.

- The FONC of k_{t+1} is $\lambda_{2,t} = \beta \mathbf{E}_t \left\{ \lambda_{1,t+1} r_{t+1} + \lambda_{2,t+1}(1-\delta) \right\}$, where $\lambda_{2,t}$ is the Lagrange multiplier attached to the law of motion of capital.

- The FONC is $q_t \lambda_{1,t} = \beta \mathbf{E}_t \left\{ \lambda_{1,t+1} \left[r_{t+1} + q_{t+1}(1-\delta) \right] \right\}$, where Tobin's q is $q_t = \dfrac{\lambda_{2,t}}{\lambda_{1,t}}$.

- Combine the FONCs to find an arbitrage condition on the real returns on k_{t+1} and B_t

$$\mathbf{E}_t \left\{ \lambda_{1,t+1} \left(\left[\frac{r_{t+1}}{q_{t+1}} + (1-\delta) \right] \frac{q_{t+1}}{q_t} - \frac{R_{B,t}}{\pi_{t+1}} \right) \right\} = \frac{\varrho_t}{\beta} \mathcal{W}'(b_{t+1}),$$

  in which the preference shock, ϱ_t, and $\mathcal{W}'(b_{t+1})$ generates a wedge in the arbitrage between these returns on the risky and (nearly) riskless assets.

- The right side is observationally equivalent to the external finance premium of BGG, but the wedge has two competing interpretations.
  1. A risk premium compensating the household for giving up a unit of the riskless asset, b_{t+1}, to hold more of the risky asset, k_{t+1}.
  2. A collateral constraint shock, ϱ_t, that disturbs the value of collateral, $\mathcal{W}'(b_{t+1})$.

Identifying a Sovereign Debt Demand Shock

- The FONC of B_{t+1} also yields a demand function for sovereign debt.

- Let $\mathcal{W}(B_{t+1}/P_t) = \ln \dfrac{B_{t+1}}{P_t}$ and pass the ln operator through the FONC of B_{t+1} to obtain

$$\ln \frac{B_{t+1}}{P_t} = \ln c_t - \ln \left[1 - \beta \mathbf{E}_t \left\{ \frac{R_{B,t}}{\pi_{t+1} g_{t+1}} \right\} \right] + u_t,$$

where $1/c_t = \lambda_{1,t}/\beta^t$, $g_{t+1} = c_{t+1}/c_t$, and $u_t = -\ln \varrho_t$.

- The FONC has become a demand function for sovereign debt that
  1. has a unit demand elasticity on consumption,
  2. a negative (semi-)demand elasticity on the (expected) real rate, and
  3. the preference shock, ϱ_t, is interpreted as a (potentially serially correlated) sovereign debt liquidity demand shock.
  4. See Krishnamurthy and Vissing-Jorgensen (2012, "The aggregate demand for Treasury debt," *Journal of Political Economy* 120, 233–267).

Identifying a Shock to Net Worth

- Include a depreciation shock, μ_t, in the law of motion of k_{t+1},

$$k_{t+1} = (1 - \mu_t \delta) k_t + \left[1 - \mathcal{Q}\left(x_t / x_{t-1}\right)\right] x_t,$$

- The depreciation shock, μ_t, is interpreted as shock to the marginal efficiency of investment; see Greenwood, Hercowitz, and Huffman (1988, "Investment, capacity utilization, and the real business cycle," *American Economic Review* 78, 402–417).

- Shut down the preference shock ϱ_t and add μ_t to turn the arbitrage condition into

$$\mathbf{E}_t \left\{ \lambda_{1,t+1} \left(\left[\frac{r_{t+1}}{q_{t+1}} + (1 - \mu_{t+1}\delta) \right] \frac{q_{t+1}}{q_t} - \frac{R_{B,t}}{\pi_{t+1}} \right) \right\} = \frac{1}{\beta} \mathcal{W}'\left(b_{t+1}\right).$$

- The arbitrage condition shows μ_t operates on the real return to the project or equity.

- Remember BGG equate the external finance premium with a firm's net worth.
  1. At the margin, the net worth of the firm is the return on equity valued at the change in Tobin's q $\Longrightarrow$ change in the market value of capital to its book value.
  2. The depreciation shock is observationally equivalent to a net worth disturbance.
  3. Since ϱ_t and μ_t are not risk premium, collateral constraint, liquidity, or net worth shocks, Bigio asks what the economics of these disturbances are.

BIGIO: LIQUIDITY SHOCKS

- **Liquidity**: An asset is liquid in trade if the buyer and seller are better off.
  1. $\Longrightarrow$ Liquidity shocks affect the value of assets in trade.
  2. Trade falls when liquidity shocks lower the collateral value of assets.

- Bigio's builds a version of the Kiyotaki-Moore model that has workers, savers, and entrepreneurs $\Longrightarrow$ the latter agents have projects but not sufficient resources to fund their projects.

- There is moral hazard when entrepreneurs run a project.
  1. Entrepreneurs have to invest some of their own resources in a project.
  2. The economy is inefficient because not all projects are funded $\Longrightarrow$ these projects cannot credibly commit future returns to investors today.
  3. A liquidity shock makes this problem worse by tightening collateral or external financing constraints.
  4. $\Longrightarrow$ Assets are less liquid, which indicates entrepreneurs obtain less funds from investors.
  5. The price in the market for funds is greater than the replacement cost of capital $\Longrightarrow$ Tobin's $q > 1$, which is a sign of an inefficient allocation of investment.

Bigio: Liquidity Shocks, Asset Prices, and Business Cycles

- Bigio uses nonlinear solution methods to solve his version of the KM model.
  1. The solution has two equilibria separated by a "liquidity frontier."
  2. One set of equilibria mimic the behavior of RBC models.
  3. In the other equilibria, liquidity shocks are observational equivalent to shocks to the marginal efficiency of investment.
  4. Collateral constraints bind in these equilibria $\Longrightarrow$ efficient allocations of resources or "investment wedges," which Bigio equates with Tobin's q.

- **Characterizing the Equilibria**: Along the liquidity frontier, the impact of liquidity shocks on collateral constraints varies positively with the external finance premium.
  1. Liquidity shocks have the biggest impact on the economy in states of small capital stocks or large TFP shocks.
  2. Still, the business cycle is not highly correlated with liquidity shocks.
  3. The reasoning is similar to that used to explain the limited role of investment shocks in DSGE models $\Longrightarrow$ investment in any period is small relative to the capital stock (*i.e.*, investment is rarely more than 8% of the capital stock.
  4. Investment shocks in Bigio's model have to be implausibly large to produce business cycle movements, say, in output $\Longrightarrow$ liquidity shock generate financial crises in which case Bigio's model economy nearly fails.
  5. Bigio: Will need to include other nominal and real shocks to the DGSE model to fit aggregate data of a developed economy.

Introduction: Shi (U. of Toronto, wp459, 2012)

- Two questions motivate the paper.
  1. Are exogenous shocks an important source of changes in liquidity in financial markets?
  2. Do these changes in financial market liquidity drive business cycle fluctuations?

- Employ the "liquidity shock hypothesis" to study these questions.

- The LSH consists of several steps.
  1. A rapid drop in financial market liquidity suggests asset prices fall $\Longrightarrow$ an extrinsic shock can cause the drop in liquidity.
  2. If investors use assets to back their debt, the fall in asset values produces collateral constrains that bind (more tightly).
  3. Given lower investment, real aggregate activity can decline enough to signal the start of a recession.

A KM model with Two Financial Market Frictions

- Shi studies a RBC model with two Kiyotaki–Moore (KM) style collateral constraints.
  1. A firm can sell only $\theta \in (0, 1)$ of its investment as new equity.
  2. Firms are also limited to selling no more than $\phi \in (0, 1)$ of their existing equity (*i.e.*, capital) $\Longrightarrow$ liquidity shocks tied to unexpected changes in ϕ.

- Limited Participation: There is a representative household whose members engage in separate activities during each date t.

- Shi solves the dynamic program of this household.
  1. $\Longrightarrow$ recursive competitive equilibrium.
  2. The solution yields decision rules of the household that can be used to study the KM-RBC numerically.
  3. Calibrate the KM-RBC model to "test" the LSH.
  4. $\Longrightarrow$ choices of θ and ϕ conditioned on data.

The KM model with Two Financial Market Frictions is a Puzzle

- The LSH is rejected in part by Shi given his calibration.
- Let there be a large negative liquidity shock $\Longrightarrow$ a large cut in ϕ that tightens collateral constraints when a firm sells its existing equity.
    1. Real activity falls at the model's calibration.
    2. Equity prices rise instead of falling, according to the model.
    3. $\Longrightarrow$ Is the LSH a useful story of financial crises and/or business cycle fluctuations?
- Shi argues his results are robust to adding debt finance, which loosens
  the collateral constraint on firm's associated with ϕ.
    1. A shock to ϕ tightens collateral constraints that makes funding investment projects more difficult for firms.
    2. $\Longrightarrow$ Price of the liquid asset, the fraction of equity available as collateral, rises with its demand all else constant.
    3. This increase in equity prices is enough for the aggregate price of equity to rise in response to an unexpected drop in ϕ.

A DSGE model with Two Financial Frictions: The Household

- Infinite horizon economy with a continuum of households on the unit interval.
- The household consists of
  1. a unit mass of members that are ex ante identical and
  2. are hit with an *iid* shock that determines whether they are entrepreneurs with probability $\pi \in (0, 1)$ or workers with probability $1 - \pi$.
  3. Workers are endowed with nothing more than a unit of time.
  4. Entrepreneurs are endowed only with an investment project.
  5. c_E = consumption of the entrepreneur.
  6. i = entrepreneur's investment.
  7. c_W = consumption of the worker.
  8. ℓ = the worker's labor supply.

A DSGE model with Two Financial Frictions: Timing Assumptions

- Each date t is divided into household, production, investment, and consumption action subperiods.
- Households: They enter date t with capital k, equity claims s, and government bonds b (*i.e.*, liquid assets).
  1. Given all members of the household are ex ante the same, they receive equal amounts of k and shares of s and b.
  2. After π is realized, state dependent actions for c_W, ℓ, s'_W, b'_W, c_E, i, s'_E, and b'_E are given to workers and entrepreneurs, where x_{t+1} is denoted x'.
- Production technology: $y = A\mathcal{F}(k_D, \ell_D)$, where y is output, A is TFP, $\mathcal{F}(\cdot,\cdot)$ is CRS, and D denotes demand.
  1. $A \sim$ stationary first-order Markov process.
  2. Pay equity claims on k_D and worker wages after y is realized.
  3. Capital depreciates at rate $1-\sigma$, $\sigma \in (0, 1)$.
  4. Equity holdings are normalized by σ to ground future dividend payments on net capital.

A DSGE model with Two Financial Frictions: Timing Assumptions, cont.

- Investment: Entrepreneurs ask households to finance their investment projects.
  1. At date t, the entrepreneur creates a unit of new capital $\Longrightarrow i$ per unit of the single consumption good.
  2. Goods and asset markets open $\Longrightarrow$ workers and entrepreneurs trade s and b, which results in (s'_W, b'_W, s'_E, b'_E).
  3. New capital is available for production next period.
- Consumption: c_W and c_E end date t.
- The household regroups only at the beginning of date $t+1$.

A DSGE model with Two Financial Frictions: The Frictions

- Remember that a household holds onto $1 - \phi$ of the s shares of equity with which it enters date $t \Longrightarrow$ only $\phi \times \sigma \times s$ can be used to fund new projects.
- Also, the share of equity finance in one unit of investment (*i.e.*, a new project) is θ at most $\Longrightarrow$ a lower bound of $1 - \theta$ on the equity the entrepreneur's household owns per unit in a new project.
- Can motivate θ and ϕ with
  1. moral hazard by an entrepreneur limits the willingness of others to invest in new projects and
  2. adverse selection places an upper bound on amount of equity that can be sold into the market at any moment in time.
- Assume θ and ϕ are constants, but
  1. can set $\phi \sim$ stationary first-order Markov process.
  2. $\Longrightarrow$ Changes in ϕ represent movements in the liquidity of equity used to finance new projects.

A DSGE model with Two Financial Frictions: The Government

- Government spends real resources, taxes households, and issues debt.
  1. g = government spending,
  2. τ = lump sum tax revenue,
  3. p_b = the price of government bonds, and
  4. B' = of new government bonds issued at the end of date t.
- The government budget constraint is $g + B = \tau + p_b B'$.
  1. Spending plus the cost of paying off maturing bonds
  2. = tax revenue plus funds generated by selling new bonds.
  3. Assume that g and $B = B'$ are positive constants.

A DSGE model with Two Financial Frictions: Household Constraints, I

- The household is subject to constraints associated with the entrepreneur and with the worker.

- The entrepreneur faces liquidity, financing, and budget constraints.
  1. Liquidity: $s_E' \geq (1-\theta)i + (1-\phi)\sigma s$.
  2. Financing: $(r + \sigma\phi q)s + b + \theta q i \geq c_E + i + \tau$.
  3. Budget: $rs + b + (i + \sigma s)q \geq c_E + i + qs_E' + p_b b_E' + \tau$.

- The worker faces equity and budget constraints.
  1. Equity: $s_W' \geq (1-\phi)\sigma s$.
  2. Budget: $(r + \sigma q)s + b + w\ell \geq c_w + qs_W' + p_b b_W' + \tau$.

- Denote r, q, and w as the rental rate of capital, the price of a unit of equity, and the real wage, respectively.

A DSGE model with Two Financial Frictions: Household Constraints, II

- Household constraints aggregate from entrepreneurial and worker constraints using $x = \pi x_E + (1-\pi)x_W$, $x = c, s, b, s'$, and b'.

- Thus, the household budget constraint is

$$(r + \sigma q)s + b + q\pi i + (1-\pi)w\ell \geq c + \pi i + qs' + p_b b' + \tau.$$

- The *iid* process π also means that the household employs the averaging equation $x = \pi x_E + (1-\pi)x_W$ to compute c_W, s'_W, and b'_W once c, c_E, s', s'_E, b', and b'_E are known.

A DSGE model with Two Financial Frictions: The HH's Dynamic Program

- Given preferences are $\pi\mathcal{U}\left(c_E\right) + (1-\pi)\left[\mathcal{U}\left(c_W\right) - \mathcal{H}\left(\ell\right)\right]$, the household's dynamic program (DP) is

$$\mathcal{V}\left(s, b, K, A, \phi\right) = \text{Max}\left[\pi\mathcal{U}\left(c_E\right) + (1-\pi)\left[\mathcal{U}\left(c_W\right) - \mathcal{H}\left(\ell\right)\right] + \beta\mathbf{E}\left\{\mathcal{V}\left(s', b', K', A', \phi'\right)\Big| K', A', \phi'\right\}\right],$$

where $\mathcal{V}\left(s, b, K, A, \phi\right)$ is the value function.

- The household chooses c, c_E, i, ℓ, s', and b' to solve the DP s.t. the household budget constraint,

$$(r+\sigma q)s + b + q\pi i + (1-\pi)w\ell \;\geq\; c + \pi i + qs' + p_b b' + \tau,$$

the entrepreneur's financing constraint,

$$(r+\sigma\phi q)s + b + \theta q i \;\geq\; c_E + i + \tau,$$

and i, c_E, c_W, s'_W, and $b'_W \geq 0$.

A DSGE model with Two Financial Frictions: Intratemporal FONCs

- Substitute $\dfrac{c - \pi c_E}{1 - \pi}$ for c_W in household preferences and use the household's budget for c to construct the FONC w/r/t c_E, ℓ, and i

$$\frac{\partial \mathcal{V}\big(s, b, K, A, \phi\big)}{\partial c_E} = \pi\Big[\mathcal{U}'\big(c_E\big) - \mathcal{U}'\big(c_W\big)\Big] - \xi_{EFC} = 0,$$

$$\frac{\partial \mathcal{V}\big(s, b, K, A, \phi\big)}{\partial \ell} = \mathcal{U}'\big(c_W\big)w - \mathcal{H}'\big(\ell\big) = 0,$$

and

$$\frac{\partial \mathcal{V}\big(s, b, K, A, \phi\big)}{\partial i} = \mathcal{U}'\big(c_W\big)\big(q - 1\big)\pi + \big(\theta q - 1\big)\xi_{EFC} = 0,$$

where ξ_{EFC} is the Lagrange multiplier on the entrepreneur's financing constraint.

A DSGE model with Two Financial Frictions: Intratemporal Optimality

- Write the intratemporal FONCs as $\mathcal{U}'\left(c_E\right) - \mathcal{U}'\left(c_W\right) = \frac{\xi_{EFC}}{\pi}$, $w = \frac{\mathcal{H}'\left(\ell\right)}{\mathcal{U}'\left(c_W\right)}$, and $q - 1 = (1 - \theta q)\frac{\xi_{EFC}}{\pi\mathcal{U}'\left(c_W\right)}$.
- The second intratemporal FONC sets the real wage to the marginal rate of substitution between work and consumption, $\Longrightarrow$ labor supply schedule.
- Define $\lambda_{EFC} \equiv \frac{\xi_{EFC}}{\pi\mathcal{U}'\left(c_W\right)}$, which yield the optimality conditions

$$\begin{aligned} \mathcal{U}'\left(c_E\right) &= \left(1 + \lambda_{EFC}\right)\mathcal{U}'\left(c_W\right), \\ \text{and} \quad q &= 1 + (1 - \theta q)\lambda_{EFC} \;\; \text{if} \;\; 0 < i, \lambda_{EFC}. \end{aligned}$$

- These optimality conditions equate
  1. the MU of entrepreneurs and workers by scaling the "discounted by marginal utility" Lagrange multiplier $\lambda_{EFC} \Longrightarrow \mathcal{U}'\left(c_E\right) > \mathcal{U}'\left(c_W\right)$, and
  2. the price of equity to one plus the "price" of a unit of i evaluated at the "discounted by marginal utility" shadow price of a unit of additional financing for the entrepreneur $\Longrightarrow$ Tobin's q and $q > 1$.

A DSGE model with Two Financial Frictions: Intratemporal Economics

- Since $\mathcal{U}'\left(c_E\right) - \mathcal{U}'\left(c_W\right) > 0$, $c_E < c_W \implies$ incomplete risk sharing.
  1. Given preferences and the two collateral constraints bind,
  2. the entrepreneur invests, $i > 0$, instead of consuming.
- The "value" of transferring a unit of MU at the equity price from the worker to the entrepreneur is $\dfrac{1-\theta q}{1-\theta}$, or the entrepreneur's leverage in a project.
- At market prices and an interior solution the household's risk sharing arrangement between the entrepreneur and the worker is

$$\frac{\mathcal{U}'\left(c_E\right)}{\mathcal{U}'\left(c_W\right)} = \frac{1-\theta}{1-\theta q}q \implies \text{risk sharing wedge} = \frac{\text{Tobin's } q}{\text{entrepreneur's leverage}}.$$

- An interior solution exists *iff* $1 < q < \dfrac{1}{\theta}$.
  1. The price of equity is $>$ cost of replacing capital, although no resources are lost by installing investment, because the collateral constraint binds $\implies 0 < \theta$.
  2. A binding collateral constraint restricts the rate at which the household is willing to trade $\mathcal{U}'\left(c_E\right)$ for a unit of $\mathcal{U}'\left(c_W\right) < q$.

A DSGE model with Two Financial Frictions: Intertemporal Optimality

- Similarly the FONCs w/r/t b' and s' are

$$\frac{\partial \mathcal{V}\left(s, b, K, A, \phi\right)}{\partial b'} = -p_b \mathcal{U}'\left(c_W\right) + \beta \mathbf{E}\left\{ \frac{\partial \mathcal{V}\left(s', b', K', A', \phi' \,\middle|\, K', A', \phi'\right)}{\partial b'} \right\} = 0,$$

and

$$\frac{\partial \mathcal{V}\left(s, b, K, A, \phi\right)}{\partial s'} = -q \mathcal{U}'\left(c_W\right) + \beta \mathbf{E}\left\{ \frac{\partial \mathcal{V}\left(s', b', K', A', \phi' \,\middle|\, K', A', \phi'\right)}{\partial s'} \right\} = 0.$$

- The associated envelop conditions are

$$\frac{\partial \mathcal{V}\left(s, b, K, A, \phi\right)}{\partial b} = \mathcal{U}'\left(c_W\right) + \xi_{EFC},$$

and

$$\frac{\partial \mathcal{V}\left(s, b, K, A, \phi\right)}{\partial s} = \left(r + \sigma q\right)\mathcal{U}'\left(c_W\right) + \left(r + \sigma \phi q\right)\xi_{EFC}.$$

A DSGE model with Two Financial Frictions: Euler Equations

- Appealing to the Benveniste and Scheinkman (Econometrica, 1979) condition, the Euler equations of the bond and equity markets are

$$p_b = \mathbf{E}\left\{\mathcal{G}\left(c'_W, c_W\right)\left[1 + \frac{(q'-1)\pi}{1-\theta q'}\right]\right\},$$

and

$$q = \mathbf{E}\left\{\mathcal{G}\left(c'_W, c_W\right)\left(\left[1 + \frac{(q'-1)\pi}{1-\theta q'}\right]\left(r' + \sigma q'\right) - \left(1-\phi'\right)\frac{(q'-1)\pi}{1-\theta q'}\sigma q'\right)\right\}$$

where the stochastic discount factor (SDF) is $\mathcal{G}\left(c'_W, c_W\right) \equiv \beta\dfrac{\mathcal{U}'\left(c'_W\right)}{\mathcal{U}'\left(c_W\right)}$.

- Along any equilibrium path, household choices of c, c_E, i, ℓ, s', and b' have to satisfy these Euler equations, the intratemporal entrepreneur-worker risk sharing equation, and the intratemporal labor supply equation.

A DSGE model with Two Financial Frictions: Bond Market Dynamics

- The bond market Euler equation equates p_b to the SDF plus
  1. next period's "excess return" on equity, $(q' - 1)\pi$, per unit of next period's market value of the entrepreneur's share in an investment project.
  2. If expectation is $q' \longrightarrow 1$, p_b equals the SDF $\Longrightarrow$ price of a AD security.
  3. Otherwise, next period's collateral constraint is expected to bind, which inserts a wedge between the AD security's price and the price of the government bond $\Longrightarrow$ the riskless asset.
  4. Binding collateral constraints alter the composition of the household's portfolio $\Longrightarrow$ demand for and price of the riskless asset are higher than otherwise.

A DSGE model with Two Financial Frictions: Equity Market Dynamics

- The equity market Euler equation sets q equal to two forward looking components, both of which are weighted by the SDF.
- The first forward looking component is similar to the forward looking component of the bond market Euler equation,
  1. except that is multiplied by the "total return" to capital.
  2. $\Longrightarrow$ The return to capital plus the scaled price of equity.
- The second forward looking component nets
  1. the fraction of existing projects an entrepreneur finances
  2. multiplied by next period's "excess return" on equity, $(q' - 1)\pi$, per unit of next period's market value of the entrepreneur's share in an investment project,
  3. which is scaled by the equity (*i.e.*, capital) available after production, which is net of depreciation.
  4. If collateral constraints are not expected to bind next period $\Longrightarrow \theta, \phi', q' \longrightarrow 1$, q equals $r' + \sigma q'$ weighted by the SDF.
  5. Otherwise, whether q is higher or lower depends on which collateral constraint, θ or ϕ', binds more.

A DSGE model with Two Financial Frictions: Recursive Equilibrium, I

- The equilibrium is recursive.
  1. Solve for r and w using $A\mathcal{F}_1(k_D, \ell_D)$ and $A\mathcal{F}_2(k_D, \ell_D)$.
  2. Next, use the equilibrium and optimality to solve for the quantities c_E, c_W, i, and ℓ.
  3. $\Longrightarrow$ Decision rules that are functions of $[K \;\; Z]'$, $Z = [A \;\; \phi]$,
  4. where the flow of new equity into existing equity is πi and K and s have equivalent laws of motion.
  5. This information is useful for computing the equilibrium pricing functions of p_b and q.

- The equilibrium rests on there being a *compact* set $\mathcal{K}$ that contains all possible values of K and is a subset of $\mathbb{R}_+$ and a *compact* set $\mathcal{Z}$, which is a subset of $\mathbb{R}_+ \times [0, 1]$, and contains all possible values of Z.
  1. A set C_1 has all *continuous* functions that map $\mathcal{K} \times \mathcal{Z}$ into $\mathbb{R}_+$.
  2. C_2 is the set containing all *continuous* functions that map $\mathcal{K} \times [0, B] \times \mathcal{K} \times \mathcal{Z}$ into $\mathbb{R}_+$.
  3. C_3 is the set containing all *continuous* functions that map $\mathcal{K} \times [0, B] \times \mathcal{K} \times \mathcal{Z}$ into $\mathbb{R}$.

A DSGE model with Two Financial Frictions: Recursive Equilibrium, II

- Define: A *recursive competitive equilibrium* consists of asset and factor price functions (p_b, q, r, w) belonging in C_1, a household's policy functions $(i, c, c_E, \ell, s, b, s_E, b_E)$ belonging in C_2, the value function $\mathcal{V}\left(\cdot, \cdot, \cdot\right) \in C_3$, the demand for factors by final goods producers (k_D, ℓ_D), the laws of motion of K and Z, and pricing functions for p_b, q, r, w clear markets, and K and Z satisfying
  1. the solution to the household's DP problem, $\mathcal{V}\left(s, b, K, A, \phi\right)$, produces choices for $(i, c, c_E, \ell, s, b, s_E, b_E)$,
  2. $r = A\mathcal{F}_1(k_D, \ell_D)$ and $w = A\mathcal{F}_2(k_D, \ell_D)$, which are required
  3. for market clearing in goods, $c(s, b, K, Z) + \pi i(s, b, K, Z) + g = A\mathcal{F}(k_D, \ell_D)$, labor, $\ell_D = (1-\pi)\ell(s, b, K, Z)$, and capital, $k_D = K = s$,
  4. similarly, markets for the riskfree asset, $b' = b(s, b, K, Z) = B$, and equity, $s'(s, b, K, Z) = \sigma s + \pi i(s, b, K, Z)$, clear when $p_b, q \in (0, \infty)$, and
  5. the law of motion of aggregate capital matches decisions by households,
     $K' = \sigma K + \pi i(s, b, K, Z)$.

A DSGE model with Two Financial Frictions: The Equilibrium Mapping

- Existence of an equilibrium relies on the mapping $\mathcal{T}$ taking p_b, q into themselves, $\left(p_b, q\right) = \mathcal{T}\left(p_b, q\right)$.

- To show $\left(p_b, q\right) = \mathcal{T}\left(p_b, q\right)$ exists,
  1. solve r and w as functions of (ℓ, K, Z),
  2. use these functions to compute i, c, and c_W as functions of (ℓ, K, Z), which implies ℓ is a function of $(K, Z) \Longrightarrow i$, c, c_E, c_W, r, and w are functions of (K, Z),
  3. next substitute these decisions rules into the Euler equations of p_b and q to construct $\mathcal{T}\left(p_b, q\right)$ as a function of (K, Z), where the law of motion of K substitutes for K' and the first order Markov structure of Z substitutes for Z'.

- $\mathcal{T}\left(p_b, q\right)$ maps into $\mathbb{R}_+$ for any $\left(p_b, q\right) \in C_1$.
  1. C_1 contains all continuous functions that map $\mathcal{K} \times \mathcal{Z}$ into $\mathbb{R}_+ \Longrightarrow \mathcal{T}$ is a fixed point (contraction) mapping.
  2. Iterate $\mathcal{T}$ to show that from any arbitrary initial guesses for functions of $\left(p_b, q\right)$, there is convergence to equilibrium asset pricing functions.

A DSGE model with Two Financial Frictions: Intuition

- Equity prices rise in response to any disturbance in the model that tightens collateral constraints $\Longrightarrow$ a negative liquidity shock.
- This result is robust to including real and nominal frictions.
- The explanation is found in the intratemporal Euler equation
$$\frac{\mathcal{U}'\left(c_E\right)}{\mathcal{U}'\left(c_W\right)} = \frac{1-\theta}{1-\theta q}q.$$
- Since $\theta,\ \phi \in (0, 1)$, Tobin's q holds, $1 < q$, $\Longrightarrow$ incomplete risk sharing.
  1. There is excess demand for riskless and risky assets that fund projects.
  2. Asset prices rise given a shock that tighten collateral constraints.
- An open question is the impact of TFP shocks on collateral constraints.
  1. Suppose there is a "productivity" shock to an asset collateralizing debt.
  2. Is it the shock to the asset that matters for business cycle fluctuations?
  3. Or does the "productivity" shock resemble a liquidity shock given collateral constraints bind more?
- Replicating Appendix **C. Steady State, Calibration, and Computing Dynamics** with paper and pencil is a useful exercise.

Overview

- Are risk and leverage separate sources of financial shocks?
- Define leverage: purchase assets with borrowed funds.
- Leverage generates "risk" when asset return − loan rate > 0
  $\Longrightarrow$ shocks that move the spread have the potential
  to induce larger gains or losses as LTV rises.
- If changes in the asset return–loan rate spread are dominated by idiosyncratic shocks, there is the potential to hedge this risk.
- Aggregate shocks cannot be hedged $\Longrightarrow$ is (are) there investor(s) that can exploit a *LLN* to disperse the aggregate risk?
- See Kihlstrom and Laffont (1982, "A Competitive Entrepreneurial Model of a Stock Market," in McCall, J.J. (ed.), The Economics of Information and Uncertainty Chicago, IL: U. of Chicago Press).

Introduction

- BS's develop a DSGE model in which small aggregate shocks
  1. generate uncertainty about the length of a "recession."
  2. $\Longrightarrow$ Large fluctuations in real activity appear as if there are shifts in the economy's steady state.
- BS work with a type of Kiyotaki–Moore financial friction grounded on differences in two household types.
  1. There are productive or "expert" households and less productive households.
  2. Experts (households) take addresses i (j) on the (a separate) unit interval.
  3. Experts are impatient compared with those less productive.
  4. Only more patient households can accumulate debt $\Longrightarrow$ experts face borrowing constraints.
- Equilibrium dynamics exhibit nonlinearities $\Longrightarrow$ asymmetric responses to positive and negative shocks.

The BS Continuous Time DSGE Model: Technology

- Expert Technology: $y_t = ak_t$, where $t \in [0, \infty)$, k_t is efficiency units of expert capital, and a is the expert productivity.

- Household Technology: $\underline{y}_t = \underline{a}\,\underline{k}_t$, where $\underline{k}_t$ is efficiency units of household capital, $\underline{a}$ is the household productivity, and let $\underline{a} < a$.

- The laws of motion of expert and household capital are

$$\dot{k}_t = \Big[\Phi(\iota_t) - \delta\Big]k_t dt + \sigma k_t dZ_t, \;\; \delta, \sigma > 0,$$

and

$$\dot{\underline{k}}_t = \Big[\Phi(\underline{\iota}_t) - \underline{\delta}\Big]\underline{k}_t dt + \sigma \underline{k}_t dZ_t, \;\; \underline{\delta} > \delta,$$

where $\dot{k}_t$ ($\dot{\underline{k}}_t$), $\Phi(\cdot)$, ι_t ($\underline{\iota}_t$), δ ($\underline{\delta}$), and σ are the instantaneous change of k_t ($\underline{k}_t$), an investment cost of adjustment function with $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi'(\cdot) > 0$, and $\Phi''(\cdot) < 0$, the expert's (household's) investment per unit of capital, the depreciation rate, and the volatility or diffusion rate of the exogenous shock Z_t on k_t.

- $dZ_t \sim$ Brownian motion $\Longrightarrow$ Z_t is a continuous time "random walk" with increments that have permanent effects common to experts and households.

The BS Continuous Time DSGE Model: Preferences

- Preferences are risk neutral for households, $\mathbf{E}\left\{\int_0^\infty e^{-rt}\underline{c}_t dt\right\}$, and for experts, $\mathbf{E}\left\{\int_0^\infty e^{-\rho t}c_t dt\right\}$.
  1. $r < \rho \Longrightarrow$ households are more patient than experts.
  2. Still assume that $c_t > 0$ for all time t, but
  3. $\underline{c}_t$ is unrestricted $\Longrightarrow$ households lend to experts.

- Experts and households face the budget constraints

$$\dot{n}_t = \Big[x_t\dot{r}_{K,t} + (1 - x_t)rdt\Big]n_t - c_t,$$

and

$$\dot{\underline{n}}_t = \Big[\underline{x}_t\dot{\underline{r}}_{K,t} + (1 - \underline{x}_t)rdt\Big]\underline{n}_t - \underline{c}_t,$$

where n_t ($\underline{n}_t$), x_t ($\underline{x}_t$), and $r_{K,t}$ ($\underline{r}_{K,t}$) are the expert's (household's) net worth, the portfolio share held in k_t ($\underline{k}_t$) by the expert (household), and the expert's (household's) return on capital.
  1. r is the riskless return besides being the household discount rate.
  2. $\Longrightarrow 1 - x_t$ $(1 - \underline{x}_t)$ is the portfolio share held by the expert (household) in the riskless asset.

The BS Continuous Time DSGE Model: Bounds on Capital's Price

- Experts' impatience is an incentive for them to set $c_0 = n_0$.
  1. $\Longrightarrow$ Experts acquire $k_t,\ t > 0$, by selling equity to households.
  2. Experts are leveraged if $x_t \geq 1 \Longrightarrow$ borrow to finance k_t.
  3. Otherwise, $n_t > 0$ for $k_t > 0,\ t > 0 \Longrightarrow$ if $n_t = 0,\ x_t = 0$.

- Without real, nominal, or financial frictions, the price of capital is bounded from above and below by
  1. $\overline{q} = \text{Max}_{\iota}\, \dfrac{a - \iota}{r - [\Phi(\iota) - \delta]}$, when $k_t > 0 \Longrightarrow$ the price of the riskless asset net of adjustment costs grossed up by output per unit of capital net of investment; and
  2. $\underline{q} = \text{Max}_{\underline{\iota}}\, \dfrac{\underline{a} - \underline{\iota}}{r - [\Phi(\underline{\iota}) - \underline{\delta}]}$, when $n_t = 0$ and $k_t = 0$.
  3. The capital market is more liquid the larger is $\overline{q} - \underline{q}$.

- The Gordon dividend growth model generates the lower and upper bounds on $q \Longrightarrow$ today's q equals $\dfrac{\text{tomorrow's dividend}}{r - \text{constant dividend growth rate}}$; for example see Barsky & DeLong (QJE, 1993).

The BS Continuous Time DSGE Model: Capital Market Assumptions

- Experts hold all the equity tied to their capital $\Longrightarrow$ only experts' riskfree assets (*i.e.*, debt) trade.
- Also experts deleverage when $n_t = 0 \Longrightarrow$ sell their k_t and
  1. cannot hold capital and/or consume in the future.
  2. $\Longrightarrow$ Future flow of utility is zero.
  3. Otherwise, the Miller–Modigliani theorem holds
  4. $\Longrightarrow$ irrelevance of capital structure of experts.
- The law of motion of the price of capital is conjectured to follow

$$\dot{q}_t = \mu_{q,t} q_t dt + \sigma_{q,t} q_t dZ_t,$$

where q_t is endogenous, but bounded within $[\underline{q}_t, \overline{q}_t]$, given changes in the time-varying mean $\mu_{q,t}$ and stochastic volatility (SV), $\sigma_{q,t}$, which is "risk" common to experts and households.

The BS Continuous Time DSGE Model: Value of Capital

- Solve for the equilibrium laws of motion of $q_t k_t$, $r_{K,t}$, and $\underline{r}_{K,t}$ using Itô's calculus; see
  1. Duffie (2001, Dynamic Asset Pricing Theory, Third Edition Princeton, NJ: Princeton University Press); and
  2. Malliaris and Brock (1989, Stochastic Methods in Economics and Finance, New York, NY: North Holland, Inc.).

- The instantaneous change in $q_t k_t$ is the gain the expert receives from holding capital $\Longrightarrow$ capital gain.

- Employ Itô's product rule to compute $\dot{q_t k_t}$ $\Longrightarrow$ stochastic calculus version of the chain rule = $\dot{q}_t k_t + q_t \dot{k}_t + \sigma_{q,t}\sigma_{k,t} q_t k_t dt$,

$$\dot{\mathcal{K}}_{\mathcal{G},t} \equiv \frac{(\dot{q_t k_t})}{q_t k_t} = \Big[\Phi(\iota_t) - \delta + \mu_{q,t} + \sigma_{q,t}\sigma\Big]dt + \Big(\sigma_{q,t} + \sigma\Big)dZ_t.$$

- $\dot{\mathcal{K}}_{\mathcal{G},t}$ consists of (*i*) a time-varying mean, (*ii*) a fundamental investment shock $\Longrightarrow$ exogenous "risk", and (*iii*) SV in q_t $\Longrightarrow$ endogenous "risk".

The BS Continuous Time DSGE Model: Returns to Capital

- For experts and households, a part of the return to capital is $\dot{\mathcal{K}}_{\mathcal{G},t}$.
- Those owning capital also receive an income flow or dividends.
  1. Expert dividends per unit of $k_t = a - \iota_t$.
  2. Household dividends per unit of $\underline{k}_t = \underline{a} - \underline{\iota}_t$.
- Thus, an expert's instantaneous return to a unit of capital equals dividends per unit of k_t's value + $\dot{\mathcal{K}}_{\mathcal{G},t} \Longrightarrow \dot{r}_{K,t} = \dfrac{a - \iota_t}{q_t}dt + \dot{\mathcal{K}}_{\mathcal{G},t}$ and $\underline{\dot{r}}_{K,t} = \dfrac{\underline{a} - \underline{\iota}_t}{q_t}dt + \underline{\dot{\mathcal{K}}}_{\mathcal{G},t}$.
- Expert and household dividend yields are susceptible to endogenous risk through SV in q_t.
  1. As SV rises dividend yields drop, but
  2. the return to capital can rise or fall.
  3. Are changes in dividend yields or $\mathcal{K}_{\mathcal{G},t}$ dominant?

The BS Continuous Time DSGE Model: Equilibrium

- Given initial endowments $\left\{k_{i,0}, \underline{k}_{j,0};\ i \in [0,\ 1],\ j \in (1,\ 2]\right\}$ and $\int_0^1 k_{i,0}di + \int_1^2 \underline{k}_{j,0}dj$ $= K_0$, an *equilibrium* consists of the process (of stochastic events ordered in time) $\left\{Z_t;\ t \geq 0\right\}$, $\left\{q_t,\ n_{i,t},\ \underline{n}_{j,t} \geq 0,\ k_{i,t},\ \underline{k}_{j,t} \geq 0,\ \iota_{i,t},\ \underline{\iota}_{j,t} \in \mathbb{R},\ c_{i,t} \geq 0, \underline{c}_{j,t}\right\}_{t=0}^{\infty}$,
  1. the initial conditions $n_{i,0} = q_0 k_{i,0}$ and $\underline{n}_{j,0} = q_0 \underline{k}_{j,0}$ are satisfied,
  2. given q_t, $r_{K,t}$ and $\underline{r}_{K,t}$, experts and households maximize their lifetime utility over uncertain streams of $c_{i,t}$ and $\underline{c}_{j,t}$, and
  3. capital and consumption goods markets clear, $K_t = \int_0^1 k_{i,t}di + \int_1^2 \underline{k}_{j,t}dj$, and

$$\int_0^1 (a - \iota_{i,t})k_{i,t}di + \int_1^2 (\underline{a} - \underline{\iota}_{j,t})\underline{k}_{j,t}dj \ = \ \int_0^1 c_{i,t}di + \int_1^2 \underline{c}_{j,t}dj,$$

  4. where the law of motion of aggregate capital is

$$\dot{K}_t \ = \ \left(\int_0^1 \left[\Phi(\iota_{i,t}) - \delta\right]k_{i,t}di + \int_1^2 \left[\Phi(\underline{\iota}_{j,t}) - \underline{\delta}\right]\underline{k}_{j,t}dj\right) + \sigma K_t dZ_t.$$

- The distribution of wealth that matters is, for example, the ratio of $\int_0^1 k_{i,t}di$ to K_t $\Longrightarrow$ the wealth of the ith expert or the jth household is tiny compared to the sum of all experts or households.

The BS Continuous Time DSGE Model: Solution Algorithm

- Two steps employed to construct solution.
  1. Experts and households maximize their preferences s.t. the laws of motion of their net worths $\Longrightarrow$ characterize the equilibrium laws of motion,
  2. Show the experts' share of aggregate wealth is the single state variable of these equilibrium processes $\Longrightarrow$ knowledge of this share is sufficient to compute prices and quantities.

- BS argue that they solve the liquidity shock hypothesis.
  1. An investment shock generates a fall in q_t.
  2. When households see a sufficiently large drop in q_t, they buy k_t in a fire sale expecting that experts will buy it back at a higher price in the future.

The BS Continuous Time DSGE Model: Portfolio Choice and Investment

- Experts maximize $r_{K,t}$ by optimal choice of ι_t.
  1. The problem is to minimize the cost of installing ι_t given q_t.
  2. $\text{Min}_{\iota_t}\,[\iota_t - q_t\Phi(\iota_t)] \Longrightarrow 1/q_t = \Phi'(\iota_t)$.
  3. Appeal to the inverse function theorem $\Longrightarrow \iota_t = \iota(q_t)$.
  4. Since experts and households face the same $\Phi(\cdot) \Longrightarrow \underline{\iota}_t = \iota_t$.

- However, returns on expert and household portfolios differ.

- Since $\underline{c}_{j,t} \in (-\infty,\, \infty)$, $\Longrightarrow$ no limit to household debt accumulation.
  1. *Arbitrage* demands that households earn r on their portfolios.
  2. Households receive r from holding the riskless asset.
  3. Define $\psi_t \equiv 1 - \dfrac{1}{K_t}\displaystyle\int_1^2 \underline{k}_{j,t}\,dj = \dfrac{1}{K_t}\displaystyle\int_0^1 k_{i,t}\,di$.
  4. If $\psi_t \in (0,\, 1)$, r is also the return on $\underline{k}_t \Longrightarrow r = \dot{\underline{r}}_{K,t}/dt$.
  5. Otherwise, r dominates the household's return on $\underline{k}_t$ $\Longrightarrow \psi_t = 1$ and $r > \dot{\underline{r}}_{K,t}/dt$.

The BS Continuous Time DSGE Model: Expert Portfolio Choice

- Experts leverage their accumulation of $k_{j,t}$ by borrowing from households $\Longrightarrow x_t > 1$.
- The law of motion of n_t governs optimal choice of $x_t \Longrightarrow$ at any t, trade c_t for n_t.
  1. Conjecture $c_t = \dot{\zeta}_t n_t \Longrightarrow \dot{\zeta}_t$ is the instantaneous rate of consumption from an expert's net worth.
  2. Conjecture the equilibrium "return" process on n_t is $\dfrac{\dot{\theta}_t}{\theta_t} = \mu_{\theta,t}dt + \sigma_{\theta,t}dZ_t$, where $\mu_{\theta,t}$ and $\sigma_{\theta,t}$ are undetermined coefficients.
- Define $\theta_t n_t \equiv \mathbf{E}_t\left\{\int_t^{\infty} e^{-\rho(s-t)}c_s ds\right\} \Longrightarrow \theta_t$ is the process generating time-varying "returns" on an expert's net worth or $\theta_t \geq 1$ for all t.
- The expert's value function is $\theta_t n_t dt = \text{Max}_{\left\{x_t,\, \dot{\zeta}_t\right\}} \dfrac{1}{\rho}\left[\dot{\zeta}_t n_t + \mathbf{E}_t\left\{(\dot{\theta_t n_t})\right\}\right]$, s.t.

  $\dfrac{\dot{n}_t}{n_t} = x_t \dot{r}_{K,t} + (1 - x_t)r dt - \dot{\zeta}_t$, and $x_t, \dot{\zeta}_t \geq 0$.
  1. Experts choose $\left\{x_t, \dot{\zeta}_t\right\}$ taking θ_t as given and n_t as their state variable.
  2. The value function implies that an expert cannot increase x_t by a unit at t, hold this capital forever, and be better off.
  3. $\Longrightarrow$ The transversality condition is $\lim_{j\longrightarrow\infty} \mathbf{E}_t\left\{e^{-\rho(j-t)}\theta_j n_j\right\} = 0$.
  4. The expert must sell capital in the future and consume to be better off.

The BS Continuous Time DSGE Model: Proposition 1

- The goal is to solve for the undetermined coefficients $\mu_{\theta,t}$ and $\sigma_{\theta,t}$.
  1. Apply Itô's lemma to the laws of motion of $\dot{\theta}_t$ and $\dot{n}_t \Longrightarrow \rho\theta_t n_t dt = \zeta_t n_t$ $+ \Big[\Big(\mu_{\theta,t} + (1 - x_t)r + x_t\sigma_{\theta,t}\sigma_{n,t}\Big)dt + \mathbf{E}_t\Big\{x_t\dot{r}_{K,t} - \dot{\zeta}_t + x_t\sigma_{\theta,t}dZ_t^2\Big\}\Big]\theta_t n_t$ $= (1-\theta_t)\dot{\zeta}_t n_t + \Big[\mu_{\theta,t} + (1-x_t)r + x_t\sigma_{\theta,t}(\sigma + \sigma_{n,t})\Big]\theta_t n_t dt + x_t\mathbf{E}_t\Big\{\dot{r}_{K,t}\Big\}\theta_t n_t$, where BS place x_t before the diffusion $\sigma_{\theta,t}$ under the assumption the expert does not consume the instantaneous return on $\theta_t n_t \Longrightarrow$ it is a martingale or the entire increase in wealth is placed in k_t.
  2. The expert's Bellman equation is maximized when $\theta_t \geq 1 \Longrightarrow$ arbitrage requires capital's return to be less than or equal to r because otherwise the expert would drive $\dot{\zeta}_t \longrightarrow \infty$.
  3. When $\dot{\zeta}_t = 0$ and $x_t = 0$, $\theta_t > 1 \Longrightarrow \rho - r = \mu_{\theta,t} < 0$, which restricts θ_t to drift downward $\Longrightarrow$ otherwise the incentives is to increase consumption and push the value of the expert's problem to negative infinity.
  4. If $\dot{\zeta}_t > 0$, $x_t > 0$ but $\theta_t = 1 \Longrightarrow \rho - r - \mu_{\theta,t} = x_t\Big[\mathbf{E}_t\Big\{\frac{\dot{r}_{K,t}}{dt}\Big\} - r + \sigma_{\theta,t}(\sigma + \sigma_{n,t})\Big]$ $= 0$, which implies the expert's excess return on capital, $\mathbf{E}_t\Big\{\frac{\dot{r}_{K,t}}{dt}\Big\} - r =$ the negative of volatility, $\sigma_{\theta,t}(\sigma + \sigma_{n,t})$, which generates precautionary incentives for the expert.
  5. This also implies $\mu_{\theta,t} = \rho - r$ when $\psi_t > 0$.

The BS Continuous Time DSGE Model: The Distribution of Wealth

- Define the wealth share of experts: $\eta_t \equiv \dfrac{N_t}{q_t K_t} \in (0, 1)$, where $N_t = \int_0^1 n_{i,t} di$ and $q_t K_t = N_t + \int_1^2 \underline{n}_{j,t} dj$.
- η_t also is a measure of the leverage of the "average" expert.
  1. Experts are less "collateral" constrained the higher is η_t all else equal.
  2. $\Longrightarrow$ Collateral constraints vary with changes in the ratio of the wealth of experts to the value of aggregate capital $q_t K_t$.
- Use $\dot{n}_t = \left[x_t \dot{r}_{K,t} + (1 - x_t) r dt\right] n_t - c_t$ to aggregate over all experts

$$\dot{N}_t = rN_t dt + \int_0^1 x_{i,t} n_{i,t} di \left[\dot{r}_{K,t} - r dt\right] - C_t.$$

- Since $\int_0^1 x_{i,t} n_{i,t} di = q_t \int_0^1 k_{i,t} di = q_t K_t \psi_t$ or the sum of the share of capital in expert portfolios is the total value of capital held by experts $\Longrightarrow$ the aggregate law of motion of expert wealth is

$$\dot{N}_t = rN_t dt + q_t K_t \psi_t \left[\dot{r}_{K,t} - r dt\right] - \zeta_t N_t, \ \zeta_t \equiv C_t \big/ N_t.$$

The BS Continuous Time DSGE Model: Lemma 2

- Lemma 2 characterizes the equilibrium law of motion of the single state variable, η_t, of the BS DSGE model.
- Since η_t is the ratio of N_t to $q_t K_t$,
  1. the equilibrium law of motion of the value of the aggregate capital stock has to be constructed.
  2. $\Longrightarrow \dfrac{\dot{\left(q_t K_t\right)}}{q_t K_t} \equiv \dot{\mathcal{K}}_{\mathcal{G},t} = \psi_t \dot{\mathcal{K}}_{\mathcal{G},t} + (1 - \psi_t)\dot{\underline{\mathcal{K}}}_{\mathcal{G},t}$.
- Use the laws of motions of $\dot{\mathcal{K}}_{\mathcal{G},t}$ and $\dot{\underline{\mathcal{K}}}_{\mathcal{G},t}$ to show

$$
\begin{aligned}
\frac{\dot{\left(q_t K_t\right)}}{q_t K_t} &= \psi_t \left(\left[\Phi\left(\iota(q_t)\right) - \delta + \mu_{q,t} + \sigma_{q,t}\sigma\right]dt + (\sigma_{q,t} + \sigma)dZ_t\right) \\
&\quad + (1 - \psi_t)\left(\left[\Phi\left(\iota(q_t)\right) - \underline{\delta} + \mu_{q,t} + \sigma_{q,t}\sigma\right]dt + (\sigma_{q,t} + \sigma)dZ_t\right) \\
&= \left[\Phi\left(\iota(q_t)\right) - \delta + \mu_{q,t} + \sigma_{q,t}\sigma\right]dt + (\sigma_{q,t} + \sigma)dZ_t + (1 - \psi_t)(\delta - \underline{\delta})dt \\
&= \mathcal{K}_{\mathcal{G},t} + (1 - \psi_t)(\delta - \underline{\delta})dt \\
&= \dot{r}_{K,t} - \frac{a - \iota(q_t)}{q_t}dt + (1 - \psi_t)(\delta - \underline{\delta})dt.
\end{aligned}
$$

The BS Continuous Time DSGE Model: Lemma 2, cont.

- The value of the aggregate value of capital stock enters the denominator of the equilibrium law of motion of $\eta_t \Longrightarrow$ construct $\dfrac{\left(1\big/\dot{q_t K_t}\right)}{1\big/q_t K_t}$.

  1. Apply Itó's lemma: a second-order Taylor expansion of $Y_t = f(X_t, t)$
     $\Longrightarrow dY_t = \left[f_X(X_t,t) + f_t(X_t,t) + \dfrac{1}{2} f_{xx}(X_t,t)\sigma_t^2\right]dt + f_X(X_t,t)\sigma_t dZ_t$.
  2. The result is $\left(\dfrac{\dot{1}}{q_t K_t}\right) = -\left(\dfrac{1}{q_t K_t}\right)^2 q_t\dot{K}_t + \dfrac{1}{2}(-1)(-2)\left(\dfrac{1}{q_t K_t}\right)^3\left(q_t\dot{K}_t\right)^2$,
     where $f_t(X_t, t) = 0$ and Z_t does not have a direct impact on $q_t\dot{K}_t$.

- Since $\left(q_t\dot{K}_t\big/q_t K_t\right)^2 = (\sigma_{q,t} + \sigma)^2 dt$,

$$\left(1\big/\dot{q_t K_t}\right) = \frac{1}{q_t K_t}\left[-\left(\dot{r}_{K,t} - \left[\frac{a - \iota(q_t)}{q_t} - (1-\psi_t)(\delta - \underline{\delta})\right]dt\right) + (\sigma_{q,t} + \sigma)^2 dt\right],$$

or

$$\frac{\left(1\big/\dot{q_t K_t}\right)}{1\big/q_t K_t} = -\left[\dot{r}_{K,t} - \frac{a - \iota(q_t)}{q_t}dt + (1-\psi_t)(\delta - \underline{\delta})dt - (\sigma_{q,t} + \sigma)^2 dt\right],$$

The BS Continuous Time DSGE Model: Lemma 2, cont.

- Given the equilibrium processes of $\dfrac{\dot{N}_t}{N_t}$ and $\dfrac{\left(1\big/\dot{q_t K_t}\right)}{1\big/q_t K_t}$, use Itó's product rule to construct $\dfrac{\dot{\eta}_t}{\eta_t} = \dfrac{\dot{N}_t}{N_t} + \dfrac{\left(1\big/\dot{q_t K_t}\right)}{1\big/q_t K_t} + \dfrac{\dot{N}_t}{N_t}\dfrac{\left(1\big/\dot{q_t K_t}\right)}{1\big/q_t K_t}$, which yields

$$\begin{aligned}\frac{\dot{\eta}_t}{\eta_t} &= r\,dt + \frac{\psi_t}{\eta_t}\Big[\dot{r}_{K,t} - r\,dt\Big] - \zeta_t - \Big[\dot{r}_{K,t} - \frac{a - \iota(q_t)}{q_t}dt + (1-\psi_t)(\delta - \underline{\delta})dt \\ &- (\sigma_{q,t} + \sigma)^2 dt\Big] + \frac{\psi_t}{\eta_t}(\sigma_{q,t} + \sigma)\Big[(-1)\eta_t(\sigma_{q,t} + \sigma)\Big]dt \\ &= \left(\frac{\psi_t}{\eta_t} - 1\right)\Big[\dot{r}_{K,t} - r\,dt - (\sigma_{q,t} + \sigma)^2 dt\Big] \\ &\qquad + \Big[\frac{a - \iota(q_t)}{q_t} + (1-\psi_t)(\underline{\delta} - \delta)\Big]dt - \zeta_t,\end{aligned}$$

where the cross-product of Itó's product rule finds $\dot{r}_{K,t}$ in common across the laws of motion of η_t and $\dfrac{1}{q_t K_t}$.

The BS Continuous Time DSGE Model: Lemma 2, cont.

- Lemma 2 conjectures that the equilibrium law of motion of η_t is equivalent to the drifting diffusion process

$$\frac{\dot{\eta}_t}{\eta_t} = \mu_{\eta,t}dt + \sigma_{\eta,t}dZ_t - \zeta_t,$$

where

$$\mu_{\eta,t} = \frac{a - \iota(q_t)}{q_t} + (1 - \psi_t)(\underline{\delta} - \delta) - \sigma_{\eta,t}(\sigma + \sigma_{q,t} + \sigma_{\theta,t}),$$

and

$$\sigma_{\eta,t} = \left(\frac{\psi_t}{\eta_t} - 1\right)(\sigma_{q,t} + \sigma),$$

and $\sigma_{\theta,t}$ enters $\mu_{\eta,t}$ because $\mathbf{E}_t\left\{\frac{\dot{r}_{K,t}}{dt}\right\} - r = -\sigma_{\theta,t}(\sigma + \sigma_{n,t})$ when $\psi_t > 0$.

- A nontrivial equilibrium process for η_t requires experts to hold capital in their portfolio $\Longrightarrow$ equates excess returns on capital with volatility in portfolio process crossed with sum of volatility in capital and it price.

The BS Continuous Time DSGE Model: Proposition 2 and a Markov Equilibrium

- A Markov equilibrium consists of $q_t = q(\eta_t)$, $\theta_t = \theta(\eta_t)$, and $\psi_t = \psi(\eta_t)$.
  1. The single endogenous state variable is η_t, which is first order Markov.
  2. Given a sequence of Z_t (or its increment dZ_t) on $t \in [0, \infty)$, the equilibrium computes paths for η_t, q_t, θ_t, and ψ_t.

- Existence rests on showing the equilibrium processes of η_t, q_t, and θ_t produce ordinary differential equations for the price of capital and return to expert portfolios $\Longrightarrow$ map from η_t, q_t, θ_t, q_{t+1}, and θ_{t+1} to q_{t+2} and θ_{t+2}.

- The goal is to map from the equilibrium processes into the coefficients of these differential equations.
  1. Invoke Itó's lemma to find the volatility coefficients $\sigma_{\eta,t}$, $\sigma_{q,t}$, and $\sigma_{\theta,t}$.
  2. Match $\mu_{\eta,t}$ and $\mu_{\theta,t}$ to the drifts of η_t and q_t.

- Proposition 2 restricts
  1. the domains of $q(\eta_t)$, $\theta(\eta_t)$, and $\psi(\eta_t)$ in equilibrium to $[0, \eta^*]$, where η^* is the stochastic steady state of η_t and $\zeta_t > 0$ if $\eta_t = \eta^*$ and $\zeta_t = 0$ otherwise.
  2. $q(\cdot)$ is increasing in η and $\theta(\cdot)$ is decreasing in η, and
  3. the boundary conditions $q_0 = \underline{q}$, $\theta(\eta^*) = 1$, $q'(\eta^*) = 0$, $\theta'(\eta^*) = 0$, and $\lim_{\eta\rightarrow\infty} \theta(\eta^*) = \infty$.

The BS Continuous Time DSGE Model: The Stochastic Steady State

- There are at least two definitions of a stochastic steady state.
  1. The ergodic distribution of the endogenous variables of a model $\Longrightarrow$ the transition probabilities of η_t are time invariant and its initial condition does not matter.
  2. the location of the economy given there are no shocks between times $t - j$ and t, but shocks are expected in the future.

- BS use the latter definition to study the stochastic steady state of their DSGE model.
  1. On pages 396–397, BS state
  2. "... *our model does not set the exogenous risk σ to 0 to identify the steady state but rather fixes the volatility of macro shocks and looks for the point where the system remains stationary in the absence of shocks. Thus, the location of η^* depends on the exogenous volatility σ. It is determined indirectly through the agents' consumption and portfolio decisions, taking shocks into account.*"
  3. Since ζ_t rises as $\eta_t \longrightarrow \eta^*$, when η_t is near η^* and there is a negative shock experts consume their capital rather than selling it into the market.
  4. If η_t is much less than η^*, experts expect they will deleverage by dumping capital at fire sale prices to move risk premiums on q_t and θ_t enough to push η_t near η^* given bad shocks; see footnote 18 on page 403.

The BS Continuous Time DSGE Model: Summary I

- The financial friction is that experts cannot issue debt, which generates equilibria that are inefficient.
- The inefficiencies, which coincide with $\psi_t > 0$, are that
  1. direct impact of experts not optimally smoothing consumption,
  2. capital is misallocated, especially when experts are not highly leveraged $\Longrightarrow \eta_t$ is low or $\psi_t \in (0, 1)$, and
  3. a frictionless economy generates more investment, $\iota(\overline{q}_t) > \iota(q)$.
  4. BS emphasize that the degree of inefficiencies in the economy is inversely related to η.
- Proposition 2 implies that at the steady state of η^* there are no price effects of shock because of the boundary condition $q'(\eta^*) = 0$.
  1. $\Longrightarrow$ A financial crisis relies on the economy staring far from η^*.
  2. $\Longrightarrow$ The economy is in a state of "systemic risk," according to BS.
  3. However, whether the economy can produce a "crisis" starting from a state of systemic risk depends on the calibration.

The BS Continuous Time DSGE Model: Summary II

- BS also argue there DSGE model solves the "volatility paradox".
  1. As the volatility of Z_t, σ, becomes small, the economy still suffers from endogenous risk in q_t, $\sigma_{q,t}$.
  2. Part of the solution is that $\sigma_{q,t}$ responds little to changes in σ.
  3. Instead, changes in $\underline{a}$ and $\underline{\delta}$ matter for $\sigma_{q,t}$ $\Longrightarrow$ impact on $\underline{q}$ and the initial condition q_0, which has $\sigma_{q,t}$ varying inversely with q_0.

- BS want a DSGE model in which small aggregate shocks
  1. generate uncertainty about the length of a "recession."
  2. $\Longrightarrow$ Large fluctuations in real activity appear as if there are shifts in the economy's steady state.

- When the economy is initially far from η^*,
  1. a negative investment shock pushes η_t lower.
  2. However, the length of time until the economy returns to η^* is a function of $\sigma_{q,t}$, which is endogenous and a function of ψ_t.