Lecture Notes on
Dynamic Optimizing Models of Money

or

How I Learned to Stop Worrying and Love Monetary Economics

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Abstract

These lectures notes are intended for use in first and second year macro and money theory courses for PhD students. First, the first section offers a short review of U.S. macro data and evidence about the response of aggregate output and prices to identified supply and monetary shocks. Next, a small scale new Keynesian model is presented to motivate the usefulness of including money in dynamic stochastic general equilibrium (DSGE) models. The remainder of the notes study DSGE models with money-in-the-utility-function (MIUF) or cash-in-advance (CIA) constraints. The notes emphasize the pros and cons of MIUF and CIA constraints in DSGE models. Implicit in the notes is that the specification of objects such as the Fisher equation are restricted by the primitives of a DSGE model and, hence, are subject to cross-equation restrictions as much as any other optimality condition.

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I. Introduction to Monetary Business Cycle Models

Real business cycle (RBC) theory was the dominate paradigm in macroeconomics in the 1990s. However, by the end of the decade, research questioned whether RBC theory produced an empirically credible propagation mechanism that matches actual observations. Further, research on the recessions of the 1980s and 1990s suggested there was a substantial nominal component driving these downturns in the U.S. economy. By the mid-1990s, empirical evidence and advances in monetary business cycle models began a new push to explain business cycle fluctuations in much the same way as Keynesian IS-LM models of the 1960s. In these new Keynesian models, the key monetary propagation mechanism involves sticky prices and nominal wages.

A problem is that without some friction private agents have no incentive to hold fiat currency in stochastic dynamic general equilibrium (DSGE) models. For example, agents have no reason to place positive and finite value on fiat currency in a RBC model. Consider a canonical one-sector growth model. Since this model satisfies the first two welfare theorems (*i.e.*, has a complete set of contingent claims markets), fiat currency has zero value. The nominal aggregate price level is non-positive in this economy. Some technology or friction must be grafted onto RBC models to provide money with a strictly positive price level. Two approaches receives the most attention. One approach is money-in-the-utility function (MIUF), which is either explicitly or implicitly almost always part of new Keynesian (NK) DSGE models. Another approach uses the transactions technology of the cash-in-advance (CIA) constraint. Both approaches to modeling money in DSGE models have strengths and weaknesses.

Before studying DSGE models with MIUF or a CIA constraint, these notes aim to motivate including money in macro models. First, there is a brief survey of the data and evidence that nominal shocks matter for business cycle fluctuations. Next, a small scale NK model is used to discuss whether money is a necessary ingredient to construct monetary models.

I.A Some Visual Evidence

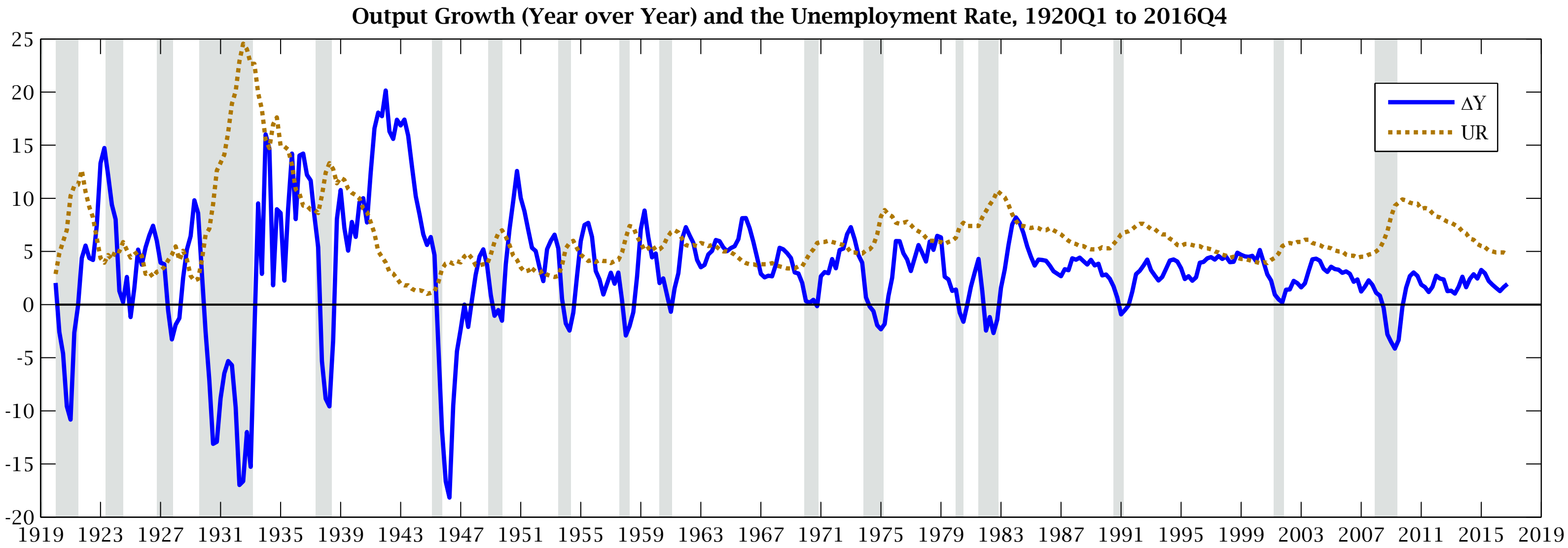
Much of modern macroeconomics is concerned with telling stories about output's response to identified productivity and monetary policy shocks. These stories are often embedded in monetary DSGE models. Macroeconomists evaluate the usefulness of these stories to describe the world by studying the fit of monetary DSGE models to the data.

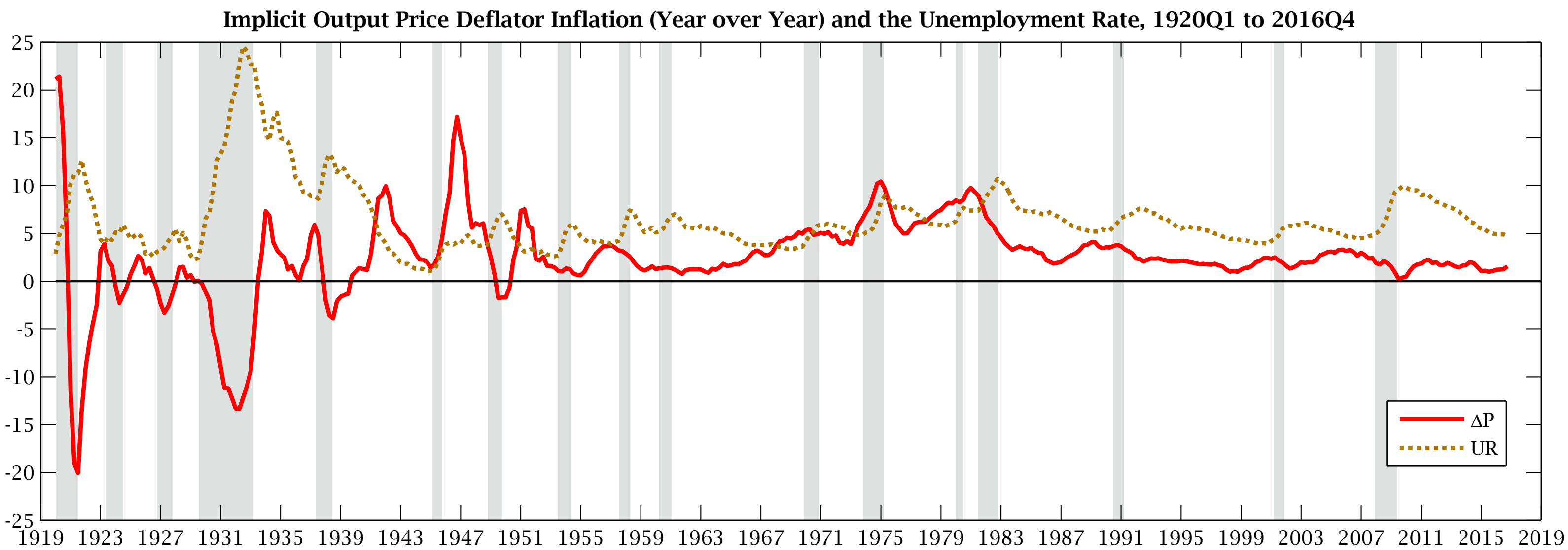
Some of this data appears in figures 1 to 6. Figure 1 plots the real side of the U.S. economy and inflation from 1920$Q4$ to 2016$Q4$. Figure 1 plots output (real GDP) growth (year over year), inflation (growth in the output price deflator, year over year), and the unemployment rate. Short and long term private and government interest rates are displayed in figures 2, 3, and 4. The latter two figures depict term, risk, and liquidity spreads using these interest rates. Term, risk, and liquidity spreads are defined as the difference between the long yield and short rate, private long yield net of the government long rate, and gap between private and government short rates, respectively. Figures 5 and 6 show (year over year) growth rates of M1, M2, M3, and the monetary base. NBER dated recessions are the vertical gray bands in figures 1 to 6.

The top panel of figure 1 contains output growth (year over year) as the solid blue line. The dotted yellow line is the unemployment rate. The deepest troughs occur in 1932 and 1946 for output growth while its greater peak is in 1942. Output growth displays substantially less variation after 1948 compared with the interwar period. There is also a drop in output growth volatility between 1984 and 2007. The unemployment rate peaks in 1933 at close to 25 percent. Since the Great Depression, the highs are little more than 10 percent in the unemployment rate during the 1981–1982 and 2007–2009 recessions. After the last recession, nearly seven years are needed for the unemployment rate to drop in half. Output growth and the unemployment rate appear to move inversely around NBER dated recessions, which suggests visual support for Okun's rule. However, figure 1 offers no evidence that output growth is either structurally causal prior to or structurally causes the unemployment rate.

The bottom panel of figure 1 presents inflation (year over year) and the unemployment rate (in levels). The solid (red) line plots inflation. The largest drop in inflation is in 1921 when the rate of deflation reaches -20 percent. During the Great Depression, there is sustained deflation that troughs at nearly -18 percent in 1932. The largest inflation spike is about 17 percent in 1946. The next peaks in inflation are about 10 percent in 1974 and 1981. Inflation is similar to output growth in that both are less volatile post–1948 and post–1984. However, these series are dissimilar in that inflation fails to peak and trough with NBER dated recessions as does output growth. Inflation and the unemployment rate display Phillips curve-like comovement at times during the sample. However, this comovement is weak compared with the negative comovement observed for output growth and the unemployment rate.

Figure 1: U.S. Real GDP Growth, Inflation, and the Unemployment Rate





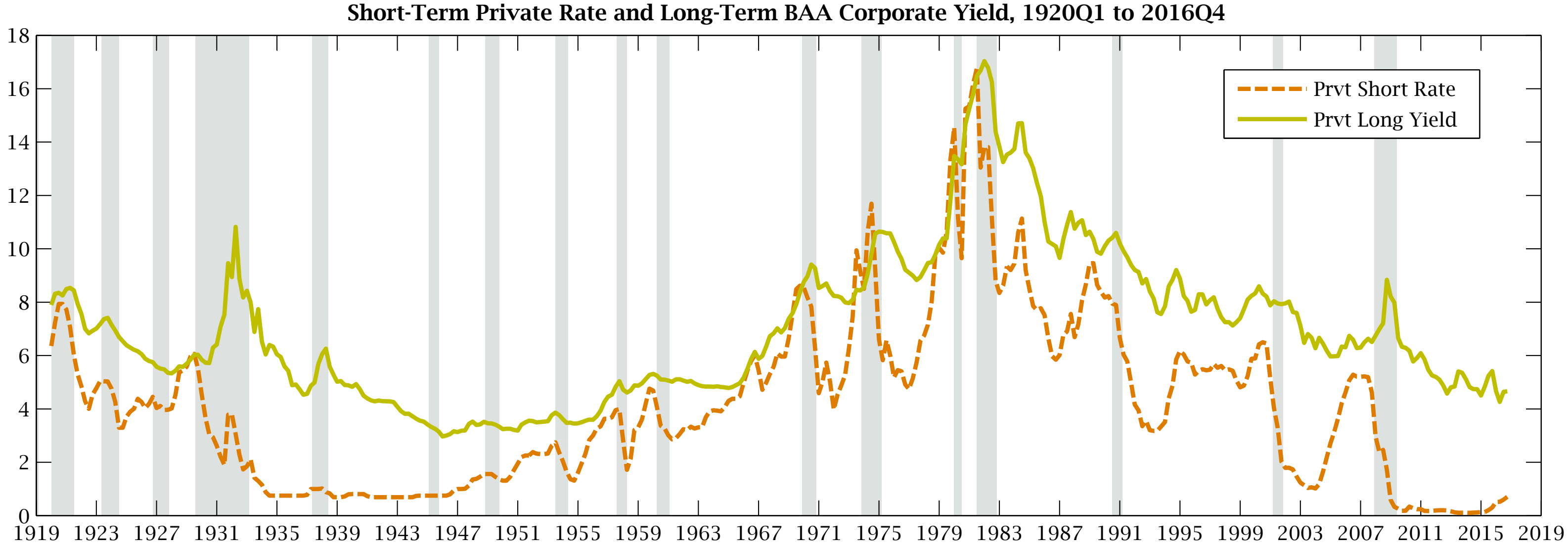
Note: The plots contain vertical gray bands that denote NBER dated recessions.

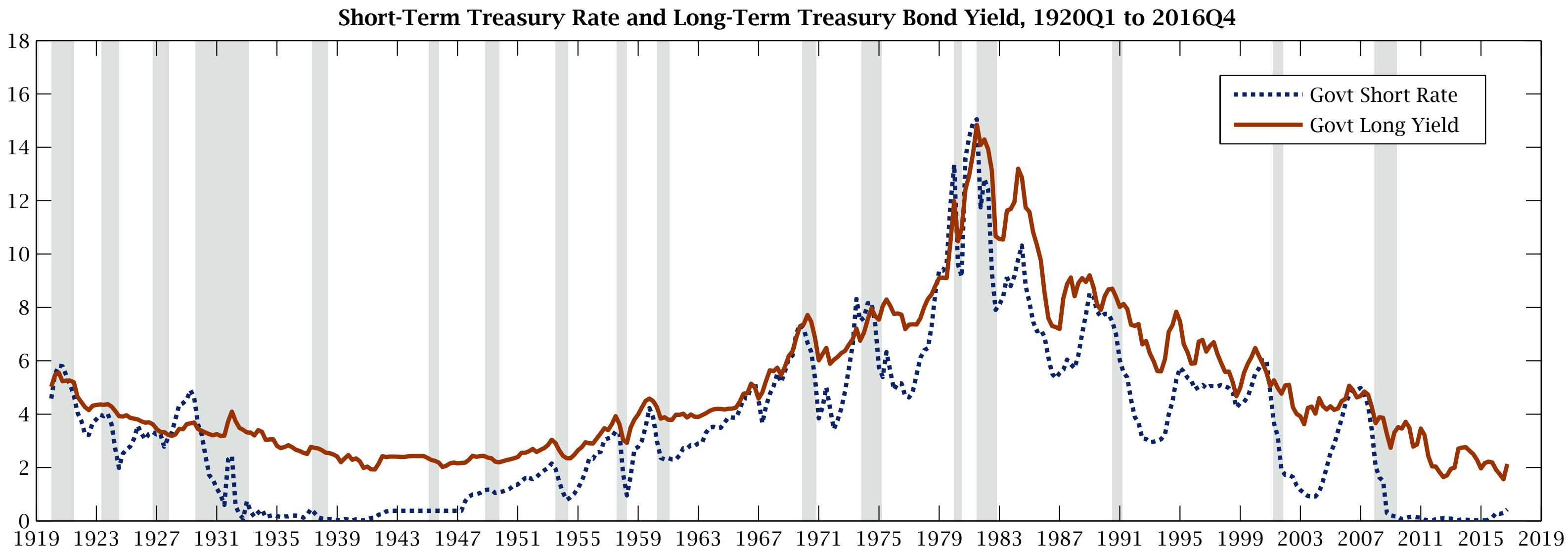
Figure 2 reports private and government nominal short term interest rates and long-term yields from 1920Q4 to 2016Q4. The private (government) short term interest rate and long term yield appear in the top (bottom) panel of figure 2. The solid lines are private (light green) and government (brick) long term interest rates in the top and bottom panels of figure 2. In the top and bottom panels of figure 2, the private short and government rates are the dashed (tan) and dotted (blue) lines.

The interest rate plots in figure 2 reveal that short and long rates fell from the beginning of the sample to 1948, except for spikes around the time of the Great Depression. From 1948 to the early 1980s interest rates increased. However, private and government short rates display cycles associated with NBER dated recessions. These fluctuations in short rates begin with the 1953–1954 recession. Short and long term rates peak around the 1981–1982 recession at more then 16 (14) percent for returns on private (government) securities. Subsequently, there is steady drop in private and government short and long rates to the end of the sample. Also, the cycles in private short and government rates become more pronounced post-1980. During the same period, business cycle-like behavior also appear in private long and government yields. Another important feature of the bottom panel of figure 2 is that there are two extended periods of near zero short term government rates during the sample. Although low from 2008 to the end of the sample, the short term government interest rate is matched by near zero returns from 1932 to 1948.

Term spreads are plotted in figure 3 from 1920Q4 to 2016Q4. The top and bottom panels of figure 3 display solid lines, which are a private (teal) and a government (light blue) spreads of yields on long maturity securities minus returns on short term securities. The private and government term spreads are similar in several ways. The term spreads are low before a NBER dated business cycle peak, rise during the recession, and peak at NBER dated recession troughs, especially post-1954. Another feature common to both term spreads is a persistent decline from 1932 to 1953. A difference is the government term spread inverts (*i.e.*, the short rate is greater than the long rate) nine times in the sample, which is more than twice as many as found for the private term spread. However, the largest inversions (in absolute value) are found in the private term spread during the 1973–1975 recession and before the recession of 1980. At the other end, the largest private term spreads occur in 1932 and 2009.

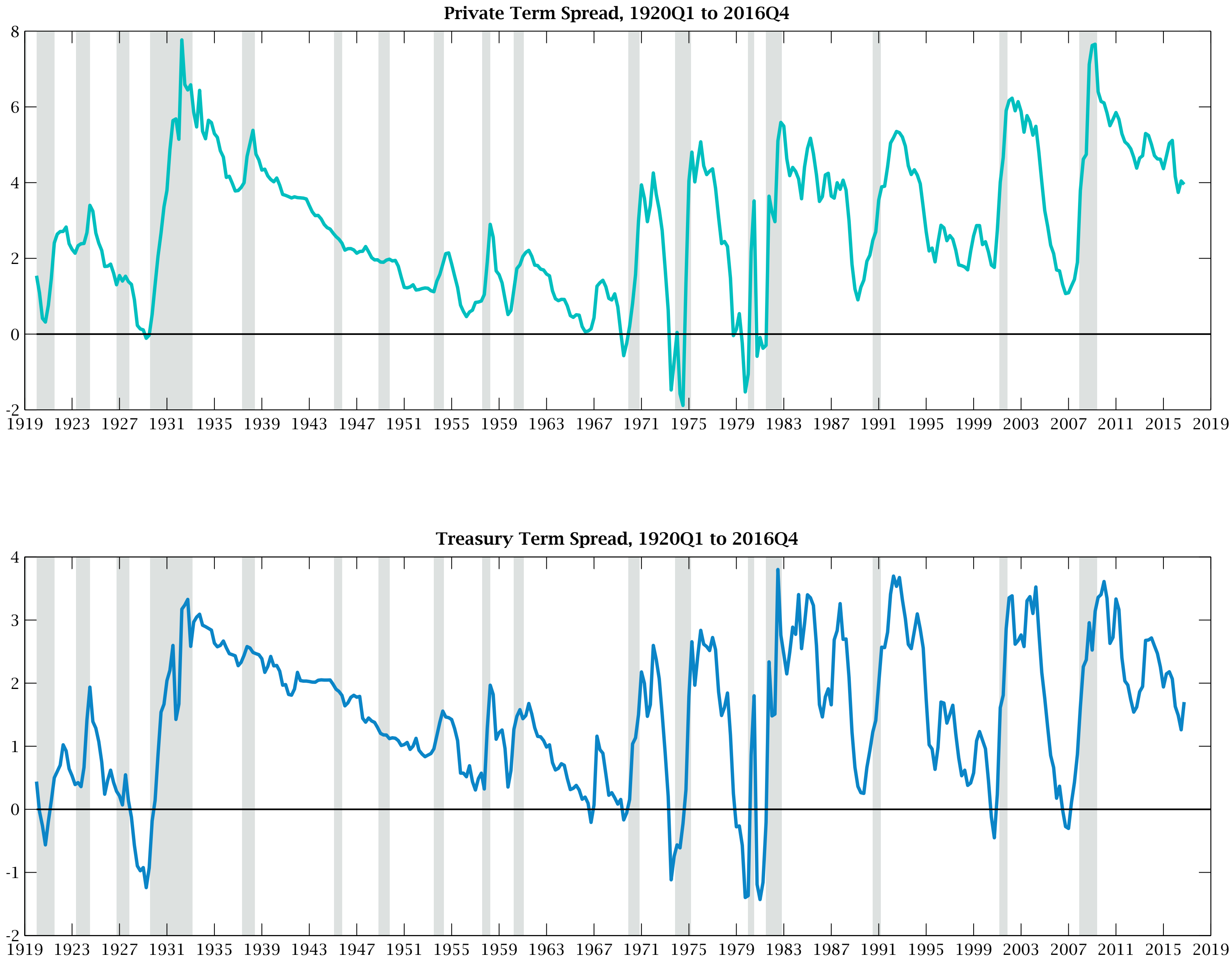
Figure 2: Short-Term and Long-Term Private and Government Interest Rates





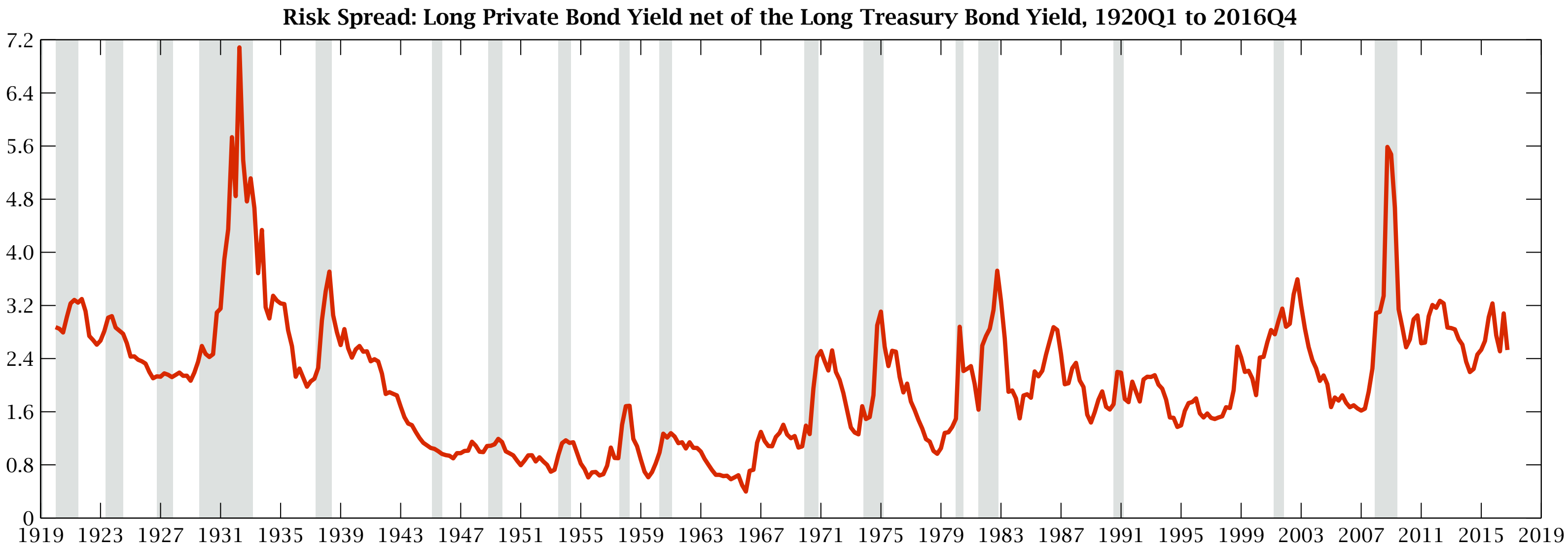
Note: The plots contain vertical gray bands that denote NBER dated recessions.

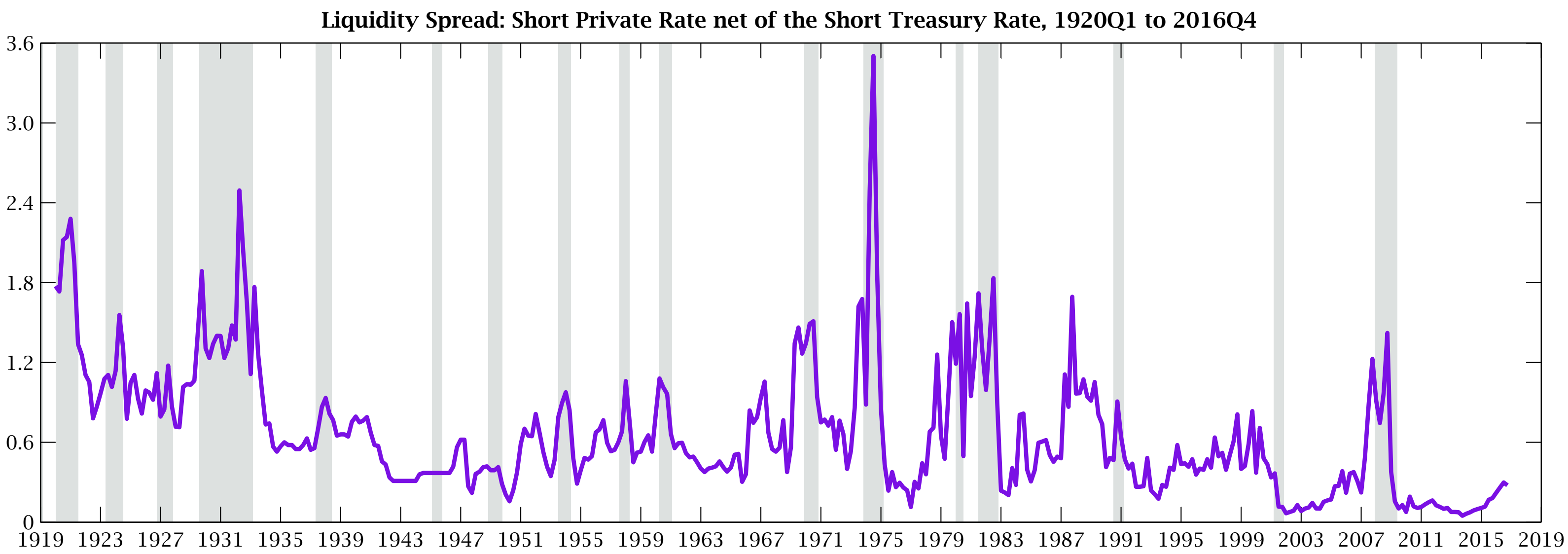
Figure 3: Private and Government Term Spreads



Note: The plots contain vertical gray bands that denote NBER dated recessions.

Figure 4: Risk and Liquidity Spreads





Note: The plots contain vertical gray bands that denote NBER dated recessions.

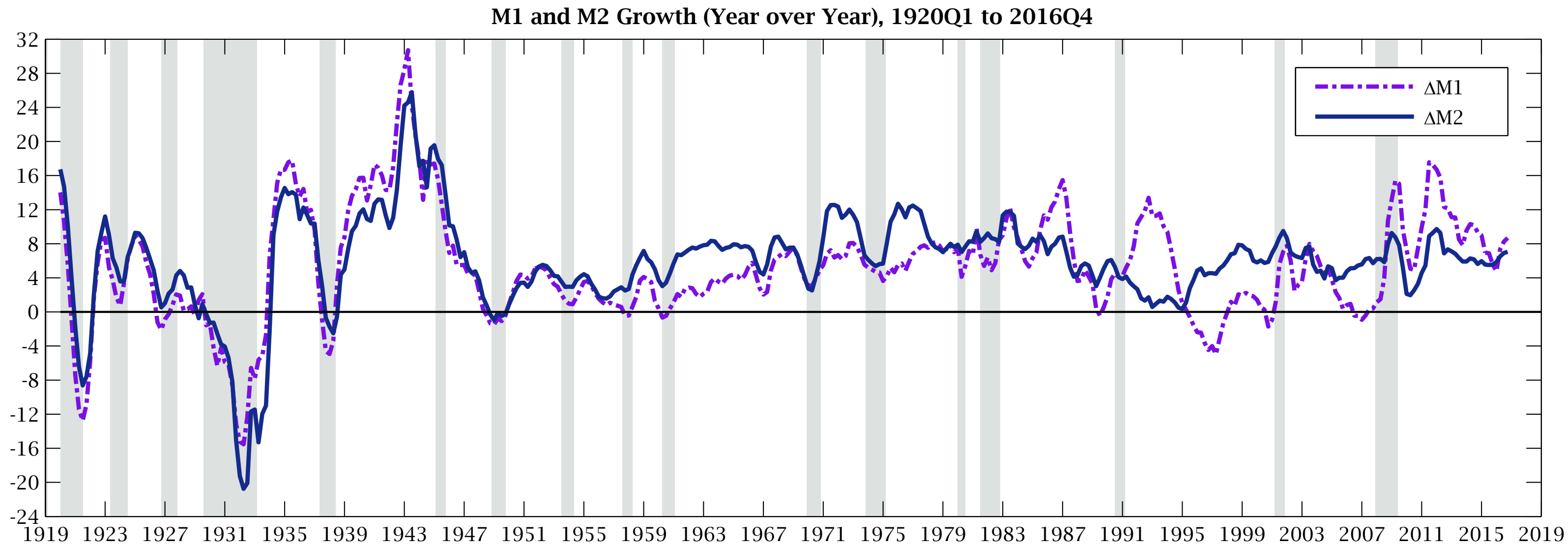
Figure 4 contains risk and liquidity spreads from 1920Q4 to 2016Q4. The risk spread is plotted as a solid (red) line in the top panel of figure 4. The most striking feature of the risk spread is that it is framed by spikes during the Great Depression and 2007–2009 recession. The former spike is greater than seven percent while the latter is about 5.6 percent. Otherwise, the risk spread is never greater than four percent during the sample. Examples are the 1937–1938 and 1981–1982 recession and in 2002. Nonetheless, the risk spread exhibits counter-cyclical business cycle comovement peaking during NBER recessions.

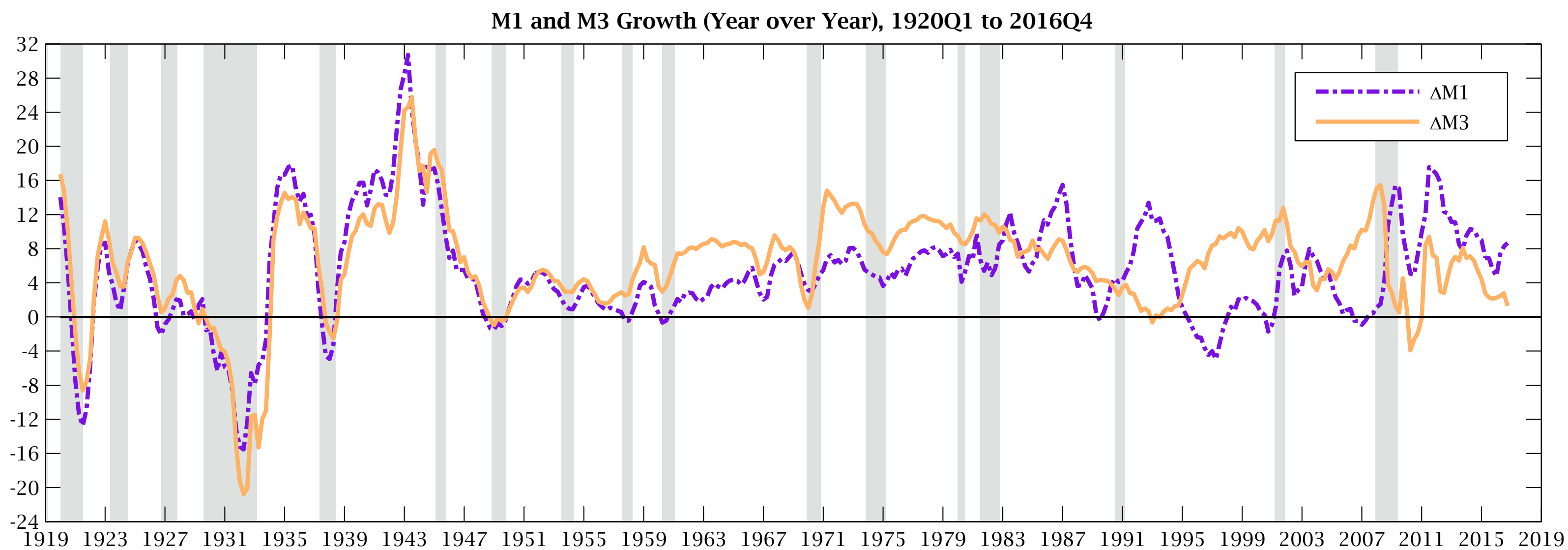
The liquidity spread is dominated by a spike during the 1973–1975 recession as shown in the bottom panel of figure 4. The next two largest peaks occur during the interwar period's recessions of 1920–1921 and 1929–1933 (*i.e.*, the Great Depression). Post–1975, the liquidity spread is high during the double dip recessions of 1980 and 1981–1982, and the stock market event of 1987. There is a spike in the liquidity spread during the 2007–2009 recession, but it is smaller than the other post–1975 spikes. The liquidity spread also peaks during NBER dated recessions similar to the risk spread.

Growth rates of U.S. monetary aggregates are found in figures 5 and 6 from 1920Q4 to 2016Q4. The top and bottom panels of figure 5 report (year over year) growth rates of M1, M2, and M3. The plots of M1 and M2, and M3 are dot-dashed (purple) and solid (dark blue) lines in the top panel and in the bottom panel M3 is the solid (orange) line. Figure 6 repeats the plots of M1 and M2 growth and adds the growth rate of the monetary base (year over year).

Figure 5 shows the volatility of the growth rates of the monetary aggregates falls after 1948. For example, M1 growth is about -12 percent in 1921, smaller than -20 percent in 1932, greater than 16 percent in 1935, and almost 32 percent in 1943. After 1948, the minimum is about -4 percent in 1997 and the maximum is 16 percent in 2011 for M1 growth. The growth paths of M2 and M3 growth display similar behavior during the sample. However, M3 growth has double digit growth for much of the 2000s. After the 2007–2009 recession, M3 growth turns negative. This makes M3 the only inside monetary aggregate to contract post–2007 while M1 and M2 growth are greater than 16 and eight percent in 2012. Nonetheless, M1, M2, and M3 fail to display consistent comovement with NBER dated recessions after 1948. The only exception is M2 growth is pro-cyclical during the 1950s, 1960s, and 1970s. This business cycle comovement disappears post–1983.

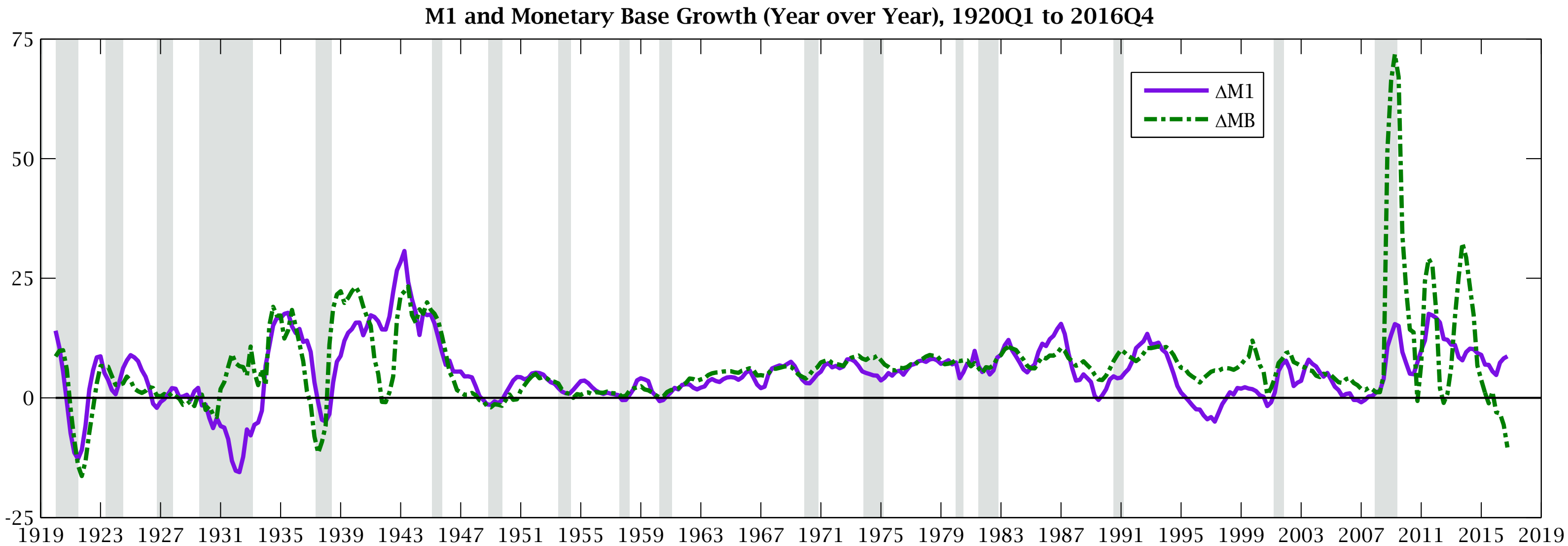
Figure 5: Growth Rates of M1, M2, and M3

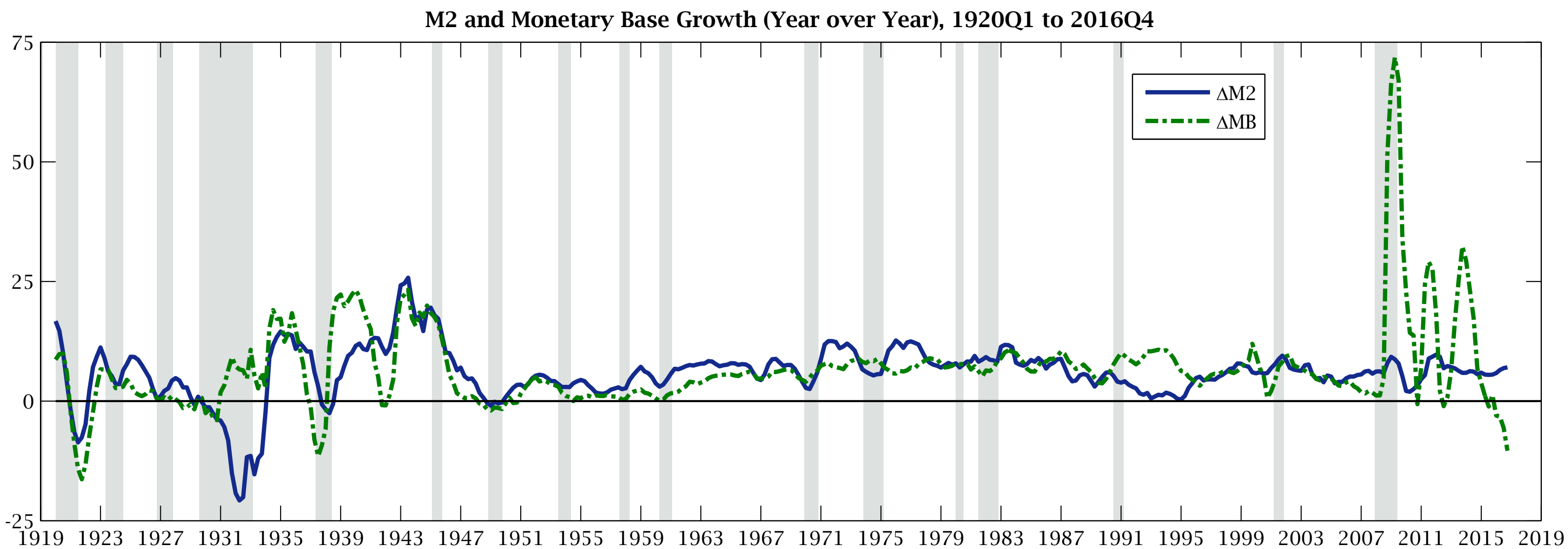




Note: The plots contain vertical gray bands that denote NBER dated recessions.

Figure 6: Growth Rates of M1, M2, and the Monetary Base





Note: The plots contain vertical gray bands that denote NBER dated recessions.

Figure 6 adds growth in the monetary base (year over year) to plots of M1 and M2 growth. The (green) dot-dash plots in the top and bottom panels of figure 6 represent monetary base growth from $1920Q4$ to $2016Q4$. Similar to the M1, M2, and M3 rates in figure 5, monetary base growth shows greater comovement with the U.S. business cycle during the interwar period than after. A reason is the monetary base was dominated by international gold flows from 1920 to 1939. After 1948, movements in reserves are the most important source of changes in the monetary base. The most eye catching part of the top and bottom panels of figure 6 is the near 75 percent spike in monetary base growth in 2009. Monetary base growth drops to about 30 percent in 2012 and 2013. The monetary base has similar growth only in the mid 1930s and from 1939 to 1944. An unresolved issue is that, although, growing post-2009, movements in M1 and M2 fail to match growth in the monetary base from 2009 to the end of the sample.

I.B Evidence from a Structural VAR

Plots of aggregate data are useful for thinking about business cycle comovement. Nevertheless, figures 1 to 6 are uninformative about structural relationships that drive business cycle fluctuations. Models built on assumptions and restrictions are needed to address questions about the responses, say, of output to productivity and monetary policy shocks.

Figures 7 and 8 have an answer to these questions using data plotted in figures 1–6. The answers are found in impulse response functions (IRFs) computed on a just-identified structural VAR. The structural VAR identifies supply and monetary policy shocks, among other disturbances. Figure 7 (8) displays IRFs of the Treasury term spread shown in the bottom panel of figure 3, PCE deflator inflation, the unemployment rate, real GDP, the effective federal funds rate, and monetary base growth with respect to a supply (monetary policy) shock from impact to a 40-quarter horizon.

The IRFs are calculated using a second-order reduced-form VAR estimated on a sample from $1960Q1$ to $2006Q4$. The supply (monetary policy) shock is identified as the orthogonalized forecast innovation of output growth (the policy rate) by imposing the ordering described in the previous paragraph on a Cholesky decomposition of the covariance matrix of the reduced-form $\text{VAR}(2)$ residuals. This ordering has the Treasury term spread taking the role of the information variable in the structural VAR. Its shock, which drives the other variables at impact, is informative about financial market expectations about inflation. This suggests, for example, a mechanism for inflation to respond to changes in inflation expectations besides the monetary policy shock. Placing inflation before the unemployment rate is consistent with a Lucas-Sargent Phillips curve while an Okun's law relation is suggested by ordering the unemployment rate before real GDP growth; see King and Watson (1994). Next, the ordering induces a monetary policy rule in which the effective fed funds rate responds to the expected inflation, current inflation, labor market, and supply shocks at impact. Monetary base growth is last in the ordering implying that demand for it reacts to unsystematic changes in the policy rate and shocks to the financial, nominal, and real sectors of the U.S. economy.

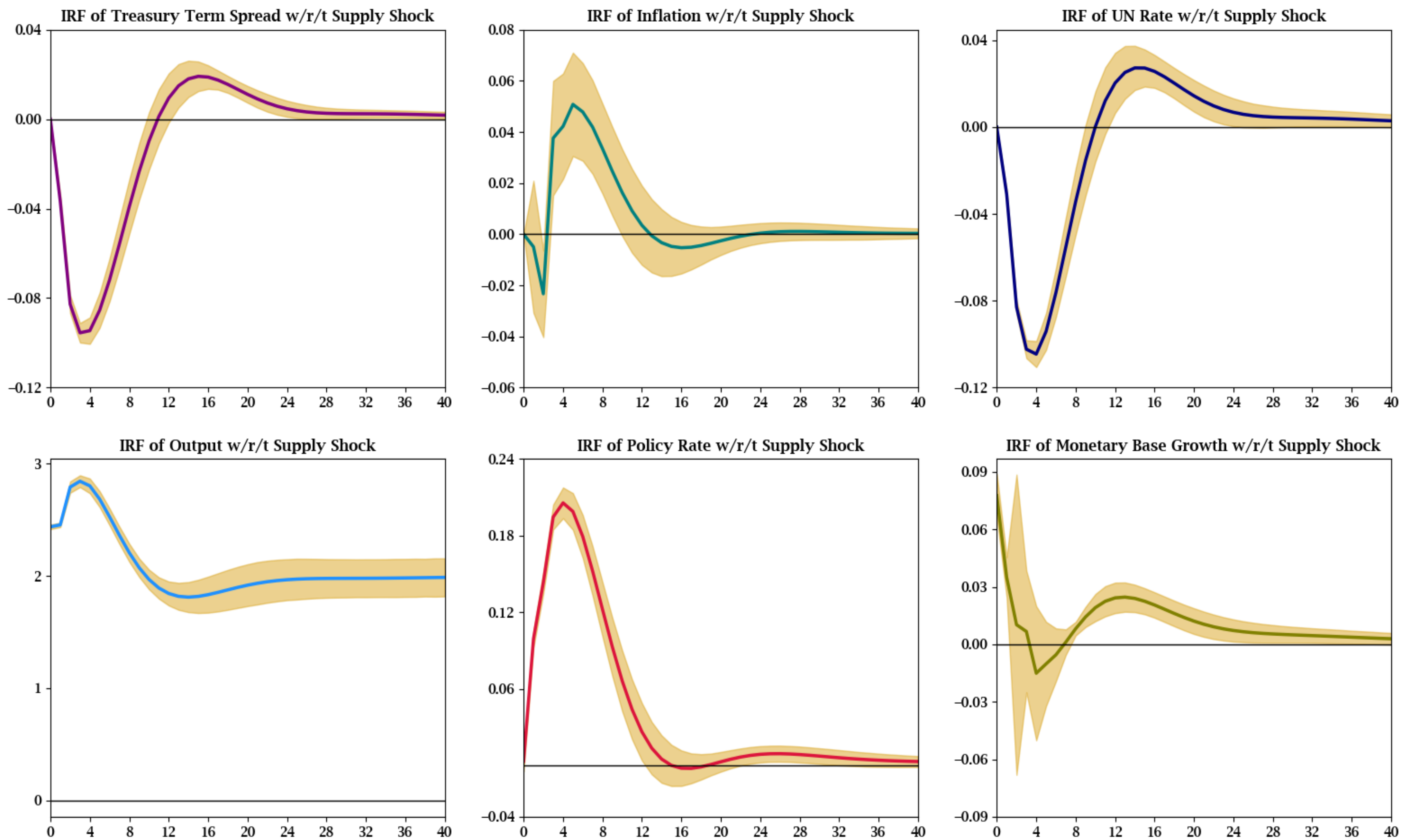
Figures 7 and 8 depict the structural IRFs as solid lines. The (goldenrod) shadings are 95 percent Bayesian uncertainty bands computed using the sup-t plugin estimator of Olea and Plagborg-Møller (2019). The uncertainty bands are produced using 20,000 draws from the reduced-form VAR(2) generated using Monte Carlo integration.

The IRF of the Treasury term spread with respect to the supply shock appears in the upper left panel of figure 7. The supply shock produces an inverted-hump shape IRF that hits a trough at three quarters. The trough is followed by a humped shaped response that peaks at 14 quarters. The IRF returns (near) to the steady state by the 6-year horizon. These responses are consistent with the plot of the Treasure term spread in the bottom panel of figure 3. The Treasury term spread often troughs (peaks) before (after) the start (end) of an NBER dated recession during the $1960Q1$–$2006Q4$ sample. The uncertainty bands of the IRF are narrow and do not cover zero from the 1-quarter to 5-year horizons.

The supply shock produces the IRF of inflation in the top middle panel of figure 7. The panel shows this IRF falls at the 1- and 2-quarter horizons. However, the uncertainty bands around these responses are wide. Only from the 3- to 10-quarter horizons is there clear evidence the response of inflation to the supply shock is strictly positive, as anticipated a priori. At longer forecast horizons, the IRF is close to zero with uncertainty containing it quarter by quarter.

The unemployment rate and the Treasury term spread share similar IRFs with respect to the supply shock as shown in the top left and right panels of figure 7. The IRF of the unemployment rate troughs at the 1-year horizon, peaks at 14 quarters, before falling toward zero by the 6-year horizon. There is also little uncertainty surrounding the IRF of the unemployment rate with respect to the supply shock from the 1-quarter to 5-year horizons.

Figure 7: Responses to an Identified Supply Shock, 1960Q1 to 2006Q4



Note: The IRFs are the solid lines from impact to 40-quarter horizon. The shadings (goldenrod) are 95 percent uncertainty bands computed using the sup-t plugin estimator of Olea and Plagborg-Møller (2019).

The bottom left panel of figure 7 displays the IRF of the level of output with respect to the supply shock. The supply shock produces a humped shaped dynamic response in output. Output is higher at impact, peaks at the 3-quarter horizon, before falling for the next eight quarters, and then levels off to the end of the IRF horizon. The uncertainty bands yield strong evidence the supply shock matters for the IRF of output from impact to the 10-year horizon.

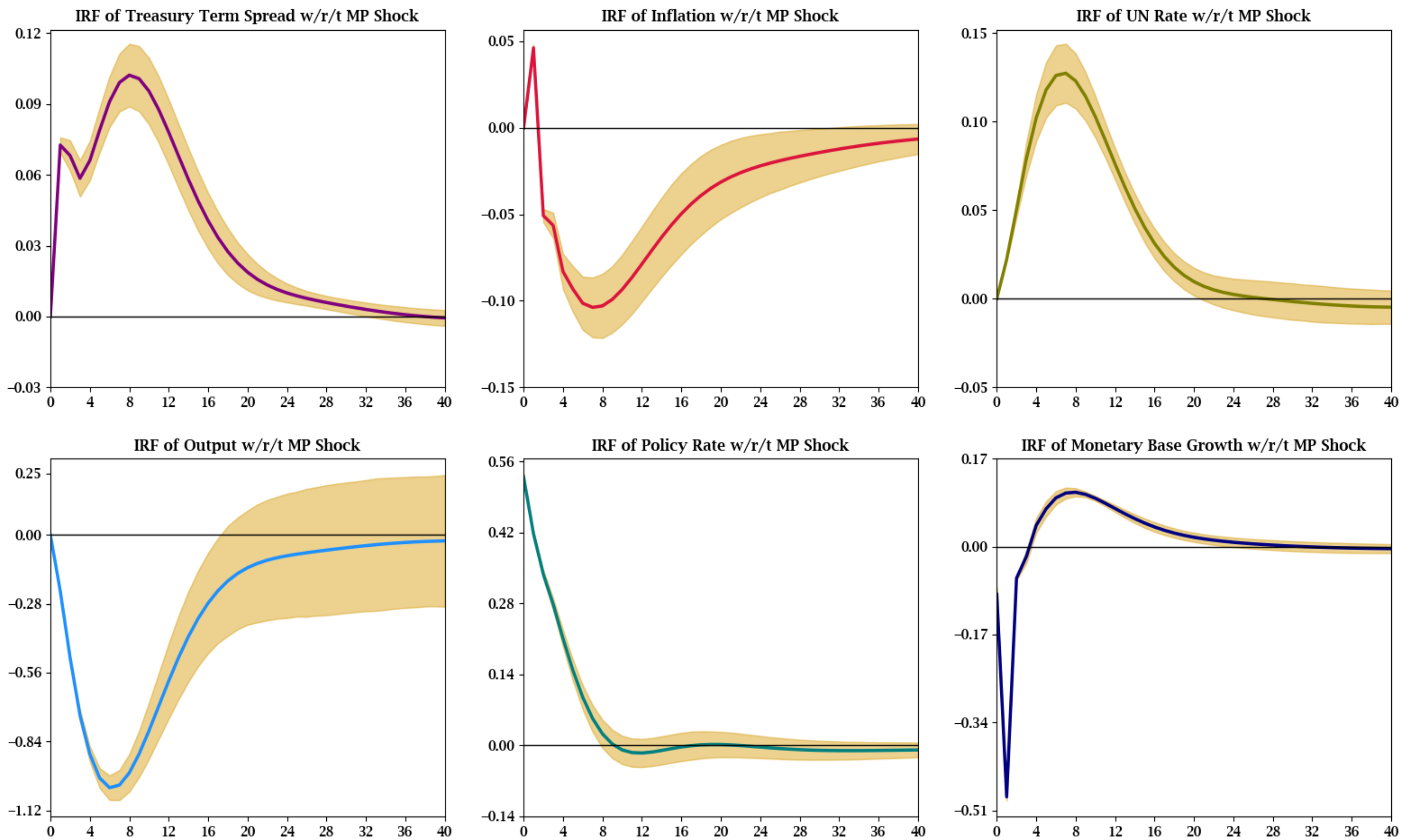
The IRF of the policy rate to the supply shock is found in the bottom middle panel of figure 7. The policy rate response to a supply shock is hump shaped from the short-run into the business cycle horizons after which it falls to steady state 14 quarters after the supply shock. The bottom middle panel of figure 7 shows the dynamic response of the policy rate to the supply shock is economically important because the uncertainty bands are narrow.

The supply shock creates the IRF of monetary base growth that appears in the bottom right panel of figure 7. The IRF, although positive at impact falls to form a V-shaped trough at the 1-year horizon. This is followed by a hump shaped path for the IRF that peaks at 14 quarters. There is substantial uncertainty around the IRF of monetary base growth with respect to the supply shock at the short- and long-horizons. Only between the 2- and 5-year horizons do the uncertainty bands not include zero quarter by quarter.

In summary, the supply shock produces responses in the Treasury term spread, PCE deflator inflation, unemployment rate, real GDP, effective fed funds rate, and monetary base growth consistent with a Treasury term structure model and a RBC model. The increase in short nominal rates dominate longer rates in the short-run, but this reverses at the business cycle horizons. Although the supply shock generates more inflation from the 1- to 3-year horizons, which suggests prices are sticky, the larger increase in short nominal rates shows real rates are higher at these horizons. Higher real rates lower the demand for the monetary base. As the real rate falls toward steady state, the demand for the monetary base rises.

A RBC model predicts the humped shape IRF of output with respect to its own shock that appears in the bottom left panel of figure 7. Transitory shocks dominate the transition path of output in the short- and medium-run while a permanent supply (*i.e.*, productivity) shock becomes important in the longer run. Cogley and Nason (1995) report similar responses of output to a permanent productivity shock and transitory government spending shock. The response of the unemployment rate to the supply shock also matches RBC theory.

Figure 8: Responses to an Identified Monetary Policy Shock, 1960Q1 to 2006Q4



Note: The IRFs are the solid lines from impact to a 40-quarter horizon. The shadings (goldenrod) are 95 percent deviation uncertainty bands computed using the sup-t plugin estimator of Olea and Plagborg-Møller (2019).

Figure 8 displays the impact of an identified monetary policy shock on the same variables. An unanticipated contractionary monetary policy shock yields an increase in the Treasury term spread, a 1-quarter increase in inflation followed by its persistent decline, a persistent decrease in real activity, and a fall in monetary base growth in the short-run that is succeeded by it rising for nearly four years. Whether these responses match the beliefs economists have about the effects a monetary policy shock has on nominal and real activity is debatable.

The contractionary monetary policy shock generates a larger increase in the yield on 10-year Treasury bonds compared with the return on 3-month Treasury bills from the short to the longer run. This explains the IRF of the Treasury term spread in the top left panel of figure 8. The middle panel of the top row of figure 8 shows inflation is higher for one quarter before falling into an inverted-hump shape in response to the contractionary monetary policy shock. Hence, this structural VAR suffers from the "price puzzle," which has the price level (or inflation) rising in reaction to a contractionary monetary policy shock; see Hanson (2004). However, the increase in inflation is short-lived lasting only one quarter after the monetary policy shock. The Treasury term spread provides information about inflation expectations in the structural VAR that limits the price puzzle only to this brief appearance.

Real activity consists of output and the unemployment rate for the VAR used to compute the IRFs reported in figure 8. The top right panel of figure 8 shows the IRF of the unemployment rate with respect to a contractionary monetary policy shock has a hump shape that peaks at a 7-quarter horizon while remaining positive for the next three years. In response to the same shock, output falls from impact to about a 6-quarter horizon before rising to zero at the 5-year horizon as shown in the bottom left panel of figure 8. The uncertainty bands of the IRF of output (the unemployment rate) cover zero from the 4- (5-) to 10-year horizon.

Monetary base growth displays two disparate responses to the monetary policy shock in the right panel of the bottom row of figure 8. The short-run response is V-shaped with a trough at the 1-quarter horizon. The IRF turns positive three quarters after the shock reaching a hump shaped peak at the 8-quarter horizon before falling to steady state 24 quarters after the shock. Slow growth in outside money is often seen as a signal for banks to raise the cost of credit. Subsequent to a contractionary monetary policy shock, faster monetary base growth indicates bank balance sheets are becoming more liquid.

I.C Monetary Policy in a New Keynesian Model that Lacks Money

New Keynesian (NK) models most often assign no role to money. Instead, the focus is on the responses of aggregate variables and inflation to an identified (and unanticipated) monetary policy shock; see Woodford (2003). Monetary policy is defined by an interest rate or Taylor rule

$$(1-\rho_R\mathbf{L})R_t \;=\; \pi_t \;+\; (1-\rho_R)\left[\kappa_\pi(\pi_t \;-\; \overline{\pi}) \;+\; \kappa_y\tilde{y}_t\right] + \varepsilon_{R,t}, \quad \varepsilon_{R,t} \sim \mathcal{N}\left(0,\, \sigma^2_{\varepsilon_R}\right), \tag{1}$$

in NK models, where $\rho_R \in (-1,\, 1)$, κ_π, $\kappa_y > 0$, $\overline{\pi}$ is the inflation target of the monetary authority and $\tilde{y}_t$ is the output gap. The monetary policy shock is $\varepsilon_{R,t}$, which represents the source of the non-systematic variation in the nominal policy rate R_t. Movements in the final targets generate systematic variation in monetary policy. The final targets are deviations of inflation, π_t, from its target, $\overline{\pi}$, and movements in the output gap, $\tilde{y}_t$. The monetary authority changes its intermediate target, the policy rate R_t, to equate π_t to $\overline{\pi}$ and set $\tilde{y}_t = 0$. The interest rate rule (1) guides the monetary authority in its quest to achieve these goals.

Money has no role in a monetary policy regime defined by the interest rate rule (1). The reason is the monetary authority supplies sufficient reserves to accommodate demand for it at R_t. However, this begs the question of the determination of equilibrium in the money market.

The monetary authority supplies money to ensure its policy rate clears the money market. Since money demand is flat with respect to R_t, understanding the incentives that give households, workers, and firms reasons to hold money is useful. One place to locate these incentives is the opportunity cost of holding fiat currency. The opportunity cost is the real rate. This turns the question into the determination of real rates in NK models.

A small scale NK model is a vehicle to study this question. Along with the interest rate rule (1), the dynamic IS schedule

$$\tilde{y}_t \;=\; \gamma_f\,\mathbf{E}_t\tilde{y}_{t+1} \;+\; \gamma_b\tilde{y}_{t-1} \;-\; \phi(R_t \;-\; \mathbf{E}_t\pi_{t+1}) \;+\; \varepsilon_{y,t}, \quad \varepsilon_{y,t} \sim \mathcal{N}\left(0,\, \sigma^2_{\varepsilon_y}\right), \tag{2}$$

and the hybrid new Keynesian Phillips curve (NKPC)

$$\pi_t \;=\; \delta_f\,\mathbf{E}_t\pi_{t+1} \;+\; \delta_b\pi_{t-1} \;-\; \lambda\tilde{y}_t \;+\; \varepsilon_{\pi,t}, \quad \varepsilon_{\pi,t} \sim \mathcal{N}\left(0,\, \sigma^2_{\varepsilon_\pi}\right), \tag{3}$$

complete the small scale NK model. Long run monetary neutrality holds if $\gamma_b = 1 - \gamma_f$. This restriction produces a vertical hybrid NKPC in the long run. Also, $\varepsilon_{y,t}$ is often interpreted as an aggregate demand shock while $\varepsilon_{\pi,t}$ is described as a markup shock.

There are five unknowns in the small scale NK model of equations (1), (2), and (3). The unknowns are $\tilde{y}_t$, π_t, R_t, $\mathbf{E}_t\tilde{y}_{t+1}$, and $\mathbf{E}_t\pi_{t+1}$. The problem is to solve for the "state" variables of the model. The state variables are $\mathbf{E}_t\tilde{y}_{t+1}$, and $\mathbf{E}_t\pi_{t+1}$. The goal is to compute these expectations. One way to do this is to marginalize the small scale NK model with respect to $\tilde{y}_t$ and π_t. The idea is to construct reduced form equations for these variables using the Taylor rule (1), dynamic IS equation (2), and hybrid-NKPC (3); see Nason and Smith (2008).

Another ingredient needed to solve the problem is a Fisher equation, which is added to the small scale NK model. An unadorned version of the Fisher equation is $r_t^E = R_t - \mathbf{E}_t\pi_{t+1}$, where r_t^E is the expected real rate. An interpretation of the Fisher equation is that it represents the demand side of the money market because r_t^E is the anticipated opportunity cost of holding money. The opportunity cost of money is the income foregone by not holding assets with a greater (real) return. Changes in r_t^E alter the demand for interest bearing assets and money as agents adjust their portfolios.

Lets show this by substituting for $\mathbf{E}_t\pi_{t+1}$ in the dynamic IS schedule (2) and the hybrid NKPC (3). The dynamic IS schedule (2) becomes

$$\tilde{y}_t \ = \ \gamma_f\,\mathbf{E}_t\tilde{y}_{t+1} \ + \ \gamma_b\tilde{y}_{t-1} \ - \ \phi r_t^E \ + \ \varepsilon_{y,t}, \tag{4}$$

which describes the equilibrium in the real economy under the Fisher equation. Solving the dynamic IS schedule (4) forward gives

$$\tilde{y}_t \ = \ \vartheta_1\tilde{y}_{t-1} \ + \ \vartheta_2\sum_{j=0}^{\infty}\vartheta_3^j\, r_{t+j}^E \ + \ \varepsilon_{y,t}, \tag{5}$$

where $\vartheta_3 \in (-1,\, 1)$ and ϑ_i, $i = 1$, 2, and 3, are functions of the forward and backward roots (or eigenvalues) of the second-order stochastic difference equation, which in turn are nonlinear functions of γ_f, γ_b, and ϕ. The solution is a AR(1) in the output gap, which is backward looking in its own lag, and forward looking in the expected present discounted value of current and future real rates; see Sargent (1987, pp. 191–192) and Hansen and Sargent (2013, pp. 95–103). The solved dynamic IS schedule (5) appears independent of nominal variables.

The Fisher equation and the solved dynamic IS (5) make the nominal side of the small scale NK model interest rate determined. Inflation becomes forward-looking in the expected real rate by substituting these equations into the hybrid NKPC (3)

$$\pi_t = \delta_f R_t + \delta_b \pi_{t-1} - \lambda\vartheta_1 \tilde{y}_{t-1} - (\delta_f + \lambda\vartheta_2) r_t^E - \lambda\vartheta_2 \sum_{j=1}^{\infty} \vartheta_3^j r_{t+j}^E + \varepsilon_{\pi,t} - \lambda\varepsilon_{y,t}, \qquad (6)$$

The revised hybrid-NKPC (6) determines π_t with predetermined variables, π_{t-1} and $\tilde{y}_{t-1}$, current nominal and expected real rates, R_t and r_t^E, and the discounted path of the expected real rate, $r_{t+1}^E, \ldots, r_{t+j}^E, \ldots$. Hence, the determination of inflation depends on expectations of future real rates. As a result, for every candidate path for the expected real rate, there is a different or multiplicity of outcomes for π_t.

Next, replace π_t and $\tilde{y}_t$ in the interest rate rule (1) using the revised hybrid-NKPC (6) and the solved dynamic IS (5). The result is the policy rate, R_t, becomes a linear function of its own lag, predetermined variables π_{t-1}, and $\tilde{y}_{t-1}$, r_t^E, and $r_{t+1}^E, \ldots, r_{t+j}^E, \ldots$. Since the interest rate rule (1) is the device the monetary authority employs to control R_t, the monetary authority must convince househlds, workers, and firms to focus their expectations on a single path for current and future real, $r_t^E, r_{t+1}^E, \ldots, r_{t+j}^E, \ldots$. Hence, the monetary authority has to choose its desired path of the expected real rate from among many to validate its monetary policy.

Monetary policy seen this way is, at least in part, about matching anticipated real returns on interest earning assets with private sector expectations. Monetary policy has to satisfy these expectations to generate a determinate (*i.e.* stable) monetary equilibrium in which the aggregate price level is strictly positive and finite, $P_t \in (0, \infty)$. However, adding several interest bearing assets to DSGE models, and NK models in particular, adds complexity that makes solving these models difficult. In an earlier tradition, money served to summarize the impact of inflation expectations on portfolio decisions in a direct and concise way in monetary models.

This suggests asking

Exercise Warming Up: These are questions about the small scale NK model.

(i) Solve the second-order difference equation (2) by pencil and paper. Show your entire work. If you need to place restriction on γ_f, γ_b, and ϕ, list these. Interpret your solution.

(ii) Does your answer to (i) affect your view of how inflation is determined in equilibrium.

(iii) What is the impact of your answers to (i) and (ii) on monetary policy under the interest rate rule (1)? Does solving the NK model of equations (1), (2), and (3) suggest that monetary policy in this economy is about r_t^E rather than R_t? Why? Does this alter your views about the usefulness of the NK model? Explain

II. Money-in-the-Utility Function

Most NK models, either explicitly or implicitly rely on MIUF to motivate the existence of nominal prices and wages. Quite literally, MIUF places real balances in the period utility of the representative agent. Real balances, m_t, equals nominal money balances, M_t, divided by the aggregate price level, P_t. Assume a perfectly inelastic labor supply, which makes period utility $\mathcal{U}(c_t, m_t)$, which is strictly concave, continuously differentiable on the positive real line, and satisfies the Inada conditions (*i.e.*, $\lim_{c\longrightarrow 0}\mathcal{U}_c(c, \cdot) = \infty$, $\lim_{c\longrightarrow\infty}\mathcal{U}_c(c, \cdot) = 0$, $\lim_{m\longrightarrow 0}\mathcal{U}_m(\cdot, m) = \infty$, and $\lim_{m\longrightarrow\infty}\mathcal{U}_m(\cdot, m) = 0$), where c_t denotes consumption, $m_t = M_t/P_t$ are real balances the household owns. The Inada conditions guarantee positive demands for consumption and real balances. Also, assume $\mathcal{U}_m(\cdot, m) \leq 0$, for all $m > \underline{m}$. The inequality states that at high levels of real balances, m, the marginal utility of real balances becomes non-positive. This is important for the price level to be finite and positive in the steady state equilibrium. For example, the MIUF specification $\mathcal{U}(c, m) = \ln c + \psi \ln m$, $\psi > 0$, violates the inequality.

II.A Deflation, Inflation, and Monetary Steady State Equilibria

The household's model is completed with the budget constraint

$$w_t + (1 + r_t)b_{t-1} + \frac{M_{t-1}}{P_t} = c_t + b_t + \frac{M_t}{P_t} - \frac{X_t}{P_t},$$

where w_t is exogenous income, b_t is the net amount of unit discount bonds the household has outstanding at the end of date t, r_t is the real return on the bonds, and X_t is a nominal lump-sum transfer (or tax) from the government. The budget constraint of the government or monetary authority is $X_t = M_t - M_{t-1}$. For the moment, assume a perfect foresight equilibrium (*i.e.*, there is no uncertainty, say, from unanticipated income shocks). The dynamic Lagrange of the household is

$$\mathcal{L}_t = \sum_{j=0}^{\infty} \beta^{t+j} \mathcal{U}(c_{t+j}, m_{t+j}) + \sum_{j=0}^{\infty} \lambda_{t+j} \Bigg[w_{t+j} + (1 + r_{t+j}) b_{t-1+j} + \frac{M_{t-1+j}}{P_{t+j}}$$

$$- c_{t+j} - b_{t+j} - \frac{M_{t+j}}{P_{t+j}} - \frac{X_{t+j}}{P_{t+j}} \Bigg],$$

where λ_t is the shadow price of another unit of income. The control variable is c_t and the state variables are b_t and m_t. The first order necessary conditions (FONCs) are

$$\beta^t \mathcal{U}_{1,t} - \lambda_t = 0,$$

$$-\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0,$$

and

$$\beta^t \frac{\mathcal{U}_{M,t}}{P_t} - \frac{\lambda_t}{P_t} + \frac{\lambda_{t+1}}{P_{t+1}} = 0.$$

Combine the first two FONCs to obtain the Euler equation for state variable b_t

$$\frac{1}{1 + r_{t+1}} = \beta \frac{\mathcal{U}_{c,t+1}}{\mathcal{U}_{c,t}}. \tag{7}$$

The Euler equation for m_t is

$$\frac{\mathcal{U}_{M,t}}{\mathcal{U}_{c,t}} = 1 - \beta \frac{\mathcal{U}_{c,t+1}}{1 + \pi_{t+1}}, \tag{8}$$

which substitutes the first FONC into the third FONC for λ_t and λ_{t+1}, where $1 + \pi_t = P_t/P_{t-1}$.

The optimality conditions (7) and (8) contain restrictions on consumption, real balances, the real rate, and inflation that must be satisfied for any equilibrium path of the economy. The real price of the bond equals the rate at which the household is willing to move consumption intertemporally, according to the Euler equation (7). The rate at which the household is willing to trade real balances for consumption intratemporally equals one minus the real return to M_t valued in utils of consumption, which describes the optimality condition (8).

The demand for money is found by eliminating $\beta\mathcal{U}_{c,t+1}$ using the optimality conditions (7) and (8). The result is

$$\frac{\mathcal{U}_{M,t}}{\mathcal{U}_{c,t}} = 1 - \frac{1}{(1 + r_{t+1})(1 + \pi_{t+1})}.$$

Since the nominal rate $R_t \approx r_t + \pi_t$, given $r_t \times \pi_t \approx 0$, the household's money demand becomes

$$\frac{\mathcal{U}_{M,t}}{\mathcal{U}_{c,t}} = \frac{R_{t+1}}{1 + R_{t+1}}.$$

The left hand side of the equality captures the marginal rate of substitution between money and consumption. Along any candidate equilibrium path, this marginal rate of substitution must equal the nominal return on the unit discount bond discounted by $1 + R_t$. The household is willing to move cash from date t to date $t + 1$ up to the point where the net benefit in utility equals the foregone discounted nominal interest. Cash (and real) balances generate an opportunity cost for the household. The ratio on the right hand side of the equality is this cost.

A central issue for monetary theory is that a steady state exists in which real balances have (strictly) positive and finite value, $m^* \in (0, \infty)$. Brock (1974, 1975) and Obstfeld and Rogoff (1983, 1986, 2021) develop the relevant results. Assume additive separability between consumption and real balances in period utility

$$\mathcal{U}(c_t, m_t) = u(c_t) + v(m_t).$$

The optimality condition (8) becomes

$$\frac{u'(c_t) - v'(m_t)}{P_t} = \beta\frac{u'(c_{t+1})}{P_{t+1}},$$

Let the growth rate of money be $1 + \mu = M_{t+1}/M_t$. Apply this to the previous expression and assume that the real side of the economy is at a steady state to produce

$$\left[u'(c^*) - v'(m_t)\right]m_t = \left[\frac{\beta u'(c^*)}{1 + \mu}\right]m_{t+1}.$$

This is a nonlinear difference equation in real balances. Lets rewrite it as

$$\mathcal{A}(m_t) = \mathcal{B}(m_{t+1}). \tag{9}$$

Without loss of generality, assume $\lim_{m \longrightarrow 0} v'(m)m = 0$. This assumption yields at least two steady state equilibria. These exists one steady state equilibrium at $m = 0$. The other generates $m^* > 0$, which implies $P^* > 0$. The former steady state equilibrium forces both sides of the difference equation (9) to be zero. This explains the need for the assumption that m plunges to zero faster than the marginal utility of real balances goes to infinity.

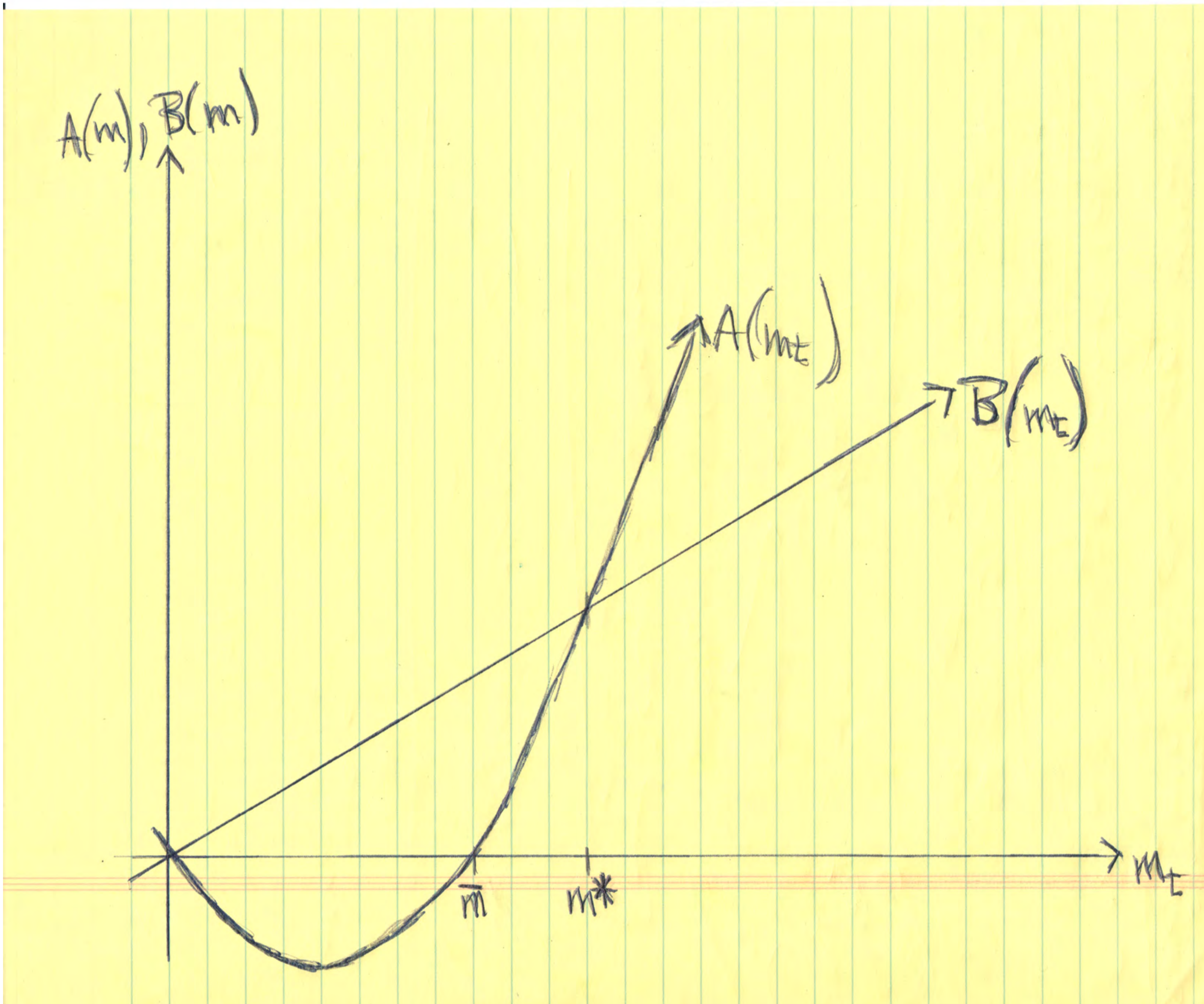
The equilibrium with positive steady state real balances is defined by the equality of (9). Figure 2.1 of Walsh (2017, p. 52) and figure 9 are visual displays of this result. However, there is a second monetary equilibrium in these figures.

Multiple monetary equilibria are created by the functions $\mathcal{A}(m_t)$ and $\mathcal{B}(m_t)$ intersecting (at least) twice. In these figures, the function $\mathcal{B}(m_{t+1})$ is the ray from the origin while $\mathcal{B}(m_t)$ is a convex function. The function $\mathcal{A}(m_t)$ initially falls as m increases because $u'(c^*) < v'(m)$ for small m. As m increases, the marginal utility of real balances falls and $\mathcal{A}(m_t)$ achieves its minimum and then begins to rise. Define m^* as the steady state level of real balances at which the functions $\mathcal{A}(m_t)$ and $\mathcal{B}(m_{t+1})$ intersect. If $m > m^*$, the transversality condition

$$\lim_{j \longrightarrow \infty} \beta^j u'(c^*)m_{t+j} = 0$$

is violated. This rules out implosive paths for the aggregate price level, $P_{t+j} \longrightarrow 0$, as $j \longrightarrow \infty$. However, there is no reason to rule out a hyperinflation. Let m decrease from m^* to zero. First, $\mathcal{A}(m_t)$ must cross the m axis to the right of the origin as m becomes small. This forces $m = \overline{m}$ and $\mathcal{A}(\overline{m}) = 0$. Hence, an initial level of m between $\overline{m}$ and m^* moves toward $\overline{m}$ and then jumps to the origin with a steady state of $m = 0$.

FIGURE 9: MULTIPLE EQUILIBRIA IN THE MIUF MODEL



This is a speculative hyperinflation because inflation is not driven by fundamentals. The price level is growing faster than the stock of nominal balances, but not because there is rapid money growth, $0 < \mu$. Instead, the hyperinflation is speculative. Households anticipate that P_t is going to infinity faster than the government can print cash. These expectations are held by a household because it anticipates other households believe the same about P_t. These beliefs have real effects. The reason is real balances appear in the utility of households. Changes in the expected future path of P_t alters the real value of future real balances. Hence, households have an incentive not to hold money because the purchasing power of money, $1/P_t$, is anticipated to fall to zero. Obstfeld and Rogoff (2021) discuss the importance of this externality for generating a speculative hyperinflation in this MIUF model, especially with respect to the critique posed by Cochrane (2011, 2019).

There are at least three ways to kill off hyper-inflations. One way is with the assumption $\lim_{m\longrightarrow 0} \mathcal{A}(m) < 0$. Although this rules out paths that converge to $m = 0$, the problem is that $m < 0$ becomes the unique steady state solution. This is clearly impossible. A second way is to rule out hyperinflation is with the restriction on the real balance component of utility

$$\lim_{m\longrightarrow 0} v'(m)m \; = \; \underline{m} \; > \; 0.$$

Hence $\lim_{m\longrightarrow 0} \mathcal{A}(m) < 0$. This is not the worse of it. Obstfeld and Rogoff (1986) show $\lim_{m\longrightarrow 0} v(m) = -\infty$. The implication is the utility derived from real balances plunges to negative infinity as real balances collapse to zero. The household can never be compensated for the drop in real balances with a finite increase in the consumption good. The definition $\mathcal{A}(m) = \left[u'(c^*) - v'(m)\right]m$ is the source of this result. The restriction on $v(m)$ forces c to rise, driving its marginal utility to zero as m increases. This is an incredible restriction on preferences, which questions the reasonableness of the MIUF approach.

A third way to rule out hyperinflation is for the government to fractional back fiat currency. Assume households believe the government will in every state of the world trade the consumption good for a unit of fiat currency. Even if the government gives the household no more than a small ϵ of the consumption good per unit of fiat currency, a lower bound is placed on the value of real balances; see Obstfeld and Rogoff (1983, pp. 683–684). Since the government is ready, willing, and able to back fiat current, households expect to obtain a strictly positive amount of consumption for fiat currency always and everywhere. Hence, the purchasing power of money is strictly positive. However, fractional backing of fiat currency suggests monetary economics cannot be separated from issues of the fiscal finance of governments.

II.B The Fisher Equation and MIUF

The MIUF approach has its costs. Care needs to be taken when real balances are placed in the household's utility function. The restrictions on the MIUF necessary to annihilate hyperinflationary equilibria is a leading example. However, MIUF is widespread because of its ease at placing a positive value on fiat currency in equilibrium

A widely used MIUF specification is

$$\mathcal{U}\left(c_t,\ \ell_t,\ \frac{M_t}{P_t}\right) \ = \ \frac{\left[c_t^{\psi}\left(\frac{M_t}{P_t}\right)^{1-\psi}\right]^{1-\alpha}}{1\ -\ \alpha}\left[\frac{\ell_t^{1-\nu}}{1\ -\ \nu}\right],$$

where ℓ is household leisure, $0 \ < \ \psi \ < \ 1$, and $0 \ < \ \alpha,\ \nu$. Remember that if any of the curvature parameters equal one, the household has log utility. Note also that MIUF imposes an externality on the household. Observe that P_t enters household preferences through real balances. Household welfare is affected by the actions of other households and the monetary authority through the determination of P_t.

The rest of the economy consists of the household's budget constraint, technology, and perfectly competitive money, bonds, and goods markets. The budget constraint is

$$y_t \ + \ (1\ -\ \delta)k_t \ + \ (1\ +\ r_t)b_t \ + \ \frac{M_t}{P_t} \ = \ c_t \ + \ k_{t+1} \ + \ b_{t+1} \ + \ \frac{M_{t+1}}{P_t} \ - \ \frac{X_t}{P_t}, \tag{10}$$

where y_t and k_t are output and the capital stock, the depreciation rate $\delta \in (0,\ 1)$. Note the change in timing of nominal balances compared with the previous monetary model. The household also owns a constant returns to scale (CRS) technology

$$y_t \ = \ k_t^{\theta}\Big[z_t\, n_t\Big]^{(1-\theta)}, \quad \theta \ \in \ (0,\ 1), \tag{11}$$

where n_t is labor input, $\ell_t = 1 - n_t$, and z_t is labor augmenting total factor productivity (TFP).

The key to the propagation mechanism of the MIUF model is the interaction of the money demand function and expected inflation effect. Lets see why. The optimality conditions with respect to b_{t+1} and M_{t+1} are

$$\left[c_t^{\psi}\left(\frac{M_t}{P_t}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_t^{1-\nu}}{c_t} = \beta\mathbf{E}_t\left\{\left[c_{t+1}^{\psi}\left(\frac{M_{t+1}}{P_{t+1}}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_{t+1}^{1-\nu}}{c_{t+1}}\left(1 + r_{t+1}\right)\right\}, \tag{12}$$

and

$$\left[c_t^{\psi}\left(\frac{M_t}{P_t}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_t^{1-\nu}}{c_t}$$

$$= \beta\mathbf{E}_t\left\{\left[c_{t+1}^{\psi}\left(\frac{M_{t+1}}{P_{t+1}}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_{t+1}^{1-\nu}}{c_{t+1}}\frac{P_t}{P_{t+1}}\left[1 + \frac{1-\psi}{\psi}\left(\frac{P_{t+1}c_{t+1}}{M_{t+1}}\right)\right]\right\}, \tag{13}$$

respectively. Bond market optimality has the usual interpretation. The household buys an additional bond to move consumption intertemporally up to the point the loss in current utility equals the discounted expected gain in utility from consuming the return to the bond.

Optimality in the money market requires the household to give up some marginal utility of consumption today to gain another unit of nominal balances up to point where the discounted expected benefit of the unit of nominal balances, which is the unit of nominal balances plus the consumption velocity of money weighted by the share of real balances to consumption in utility valued at the additional marginal utility of consumption. MIUF is the motivation for the household to value fiat currency.

The optimality conditions (12) and (13) yield the money market arbitrage condition

$$\mathbf{E}_t\left\{\left[c_{t+1}^{\psi}\left(\frac{M_{t+1}}{P_{t+1}}\right)^{1-\psi}\right]^{1-\alpha}\frac{\ell_{t+1}^{1-\nu}}{c_{t+1}}\frac{1}{1+\pi_{t+1}}\left[\frac{1-\psi}{\psi}\left(\frac{P_{t+1}c_{t+1}}{M_{t+1}}\right) - \left(r_{t+1} + \pi_{t+1}\right)\right]\right\} \approx 0, \tag{14}$$

where $r_{t+1}\pi_{t+1} \approx 0$. The arbitrage forces the marginal utility of an extra unit of nominal balances to equal the nominal return on b_{t+1}, $r_{t+1} + \pi_{t+1}$. Hence, equilibrium in the money market is driven by two factors. First, the household values real balances for its contribution to its welfare net of the real return to cash. The other factor is the opportunity cost of holding money, which is the real return to the bond.

The term in the brackets of the arbitrage condition (14) suggests a money demand function for the MIUF model. This term implies, $\ln M_t - \ln P_t = \ln c_t - (r_t + \pi_t) +$ expectation error. Money demand displays liquidity preference (*i.e.*, changes in the nominal rate alter the demand for money), a positive consumption elasticity of money demand, and an error term that is a function of the expected marginal utility of real balances and nominal rate.

The bond optimality condition (12) restricts the Fisher equation of this model. Rewrite this Euler equation as

$$1 = \mathbf{E}_t \left\{ \Gamma_{t+1} \left(\frac{M_{t+1}}{M_t} \right)^{(1-\psi)(1-\alpha)} \left(1 + \pi_{t+1} \right)^{(1-\psi)(1-\alpha)-1} \left(1 + r_{t+1} \right) \left(1 + \pi_{t+1} \right) \right\},$$

where $\Gamma_{t+1} \equiv \beta \left(\frac{c_{t+1}}{c_t} \right)^{\psi(1-\alpha)-1} \left(\frac{\ell_{t+1}}{\ell_t} \right)^{1-\nu}$, which folds the growth rate of leisure into the SDF. The previous Euler equation becomes

$$1 = \mathbf{E}_t \left\{ \Gamma_{t+1} \left(1 + \mu_{t+1} \right)^{(1-\psi)(1-\alpha)} \left(1 + \pi_{t+1} \right)^{(1-\psi)(1-\alpha)-1} \left(1 + R_{t+1} \right) \right\},$$

where $R_t = (1 + r_t)(1 + \pi_t)$ and denoting the growth rate of money as $1 + \mu_{t+1} = M_{t+1}/M_t$. Identify the inverse of Γ_t with the stochastic risk-free (real) rate, $r_{F,t+1}$ and log linearize the bond market optimality condition to produce Fisher's equation

$$\mathbf{E}_t \left\{ R_{t+1} \right\} = \mathbf{E}_t \left\{ r_{F,t+1} \right\} + \mathbf{E}_t \left\{ \pi_{t+1} \right\} + (1 - \psi)(1 - \alpha) \mathbf{E}_t \left\{ \pi_{t+1} - \mu_{t+1} \right\}. \tag{15}$$

Fisher's equation holds exactly in this model when utility is log, $\alpha = 1$, or if expected inflation equals expected money growth. Otherwise, an increase in expected inflation generates positive or negative changes in the expected nominal bond return, given the preference parameters α and ψ. Hence, the monetary propagation mechanism of the MIUF approach is either the expected inflation effect or the liquidity effect. Greater curvature in the utility function (*i.e.*, a larger α gives greater risk aversion) is required for the liquidity effect to dominate. Otherwise, the expected inflation effect drives fluctuations in the nominal return to the unit discount bond. Changes in the intertemporal opportunity cost of money matter more for movements in the expected nominal rate. In either case, movements in $\mathbf{E}_t\pi_{t+1}$ generate variation in the nominal rate independent of real factors (*i.e.*, consumption growth and leisure). This produces fluctuations in consumption because of the impact on the Euler equation of capital.

This suggests several questions about the impact of TFP and money growth on the model.

Exercise MIUF: Let the money growth shock and TFP follow

$$\mu_{t+1} = (1-\rho_\mu)\mu^* + \rho_\mu\mu_t + \eta_{t+1}, \quad \eta_{t+1} \sim \mathcal{N}\left(0, \sigma_\eta^2\right), \tag{16}$$

and

$$\ln z_{t+1} = \gamma + \ln z_t + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \mathcal{N}\left(0, \sigma_\varepsilon^2\right), \tag{17}$$

where $\rho_\mu \in (-1, 1)$ and $\gamma > 0$. Define the stationary component of the aggregate price level to be $\widetilde{P}_t = P_t A_t / M_t$.

(i) In MIUF models, does an increase in expected inflation induce the household to demand more real balances or less? Explain.

(ii) Stochastically detrend the MIUF model just presented and compute the steady state of its endogenous variables. Describe the steady state.

(iii) An economy exhibits superneutrality when changes in the money growth rate have no impact on the levels of real variables. What restrictions on the utility function parameters are necessary for this MIUF model to possess superneutrality at the steady state? [Hint: Examine optimality in the labor market.] Does it matter that leisure is non-additively separable from consumption and real balances in the utility function? Explain.

(iv) Discuss the impact of the restrictions on the preference parameters suggested by part (ii) of this question on the transition dynamics to the steady state (you do not have to linearize the model to answer the question, but it might help). Do anticipated or unanticipated movements in money growth matter for real variables along the transition path? That is, does this economy posses expectational neutrality? Explain.

(v) Discuss the role MIUF has in consumption. [Hint: Is the cost of obtaining consumption lower when the household has more real balances?]

III. Shopping Time Technology and the CIA Constraint

MIUF is a straightforward way to introduce fiat currency into a DSGE model. Since real balances provide utility, the household is given a reason to hold fiat currency. In some cases, the aggregate price level has strictly positive and finite value in equilibrium. Still this begs the question of the source(s) of the transactions services that makes real balances valuable for the household. The answer to the question is that MIUF is an indirect utility function. The task is to study the primitives that underlie this indirect utility function.

Suppose the household has to use its real balances and some of its time endowment to shop. The maintained assumptions are that it takes time to shop and the costs of shopping are inversely related to real balances. These suggest the shopping time function

$$c_t = \mathcal{C}\left(n_{c,t}, \frac{M_t}{P_t}\right), \quad \mathcal{C}_x \geq 0, \quad \mathcal{C}_{x,x} \leq 0, \quad x = n_{c,t}, \frac{M_t}{P_t},$$

where $n_{c,t}$ is the part of the household time endowment spent shopping and $1 = \ell_t + n_t + n_{c,t}$. No restriction is placed on the cross derivative of the shopping time technology, $\mathcal{C}_{n,m}$, but assume that $\mathcal{C}(\cdot, \cdot)$ satisfies the requirements of the inverse function theorem, $n_{c,t} = \mathcal{N}\left(c_t, \frac{M_t}{P_t}\right)$. Next, assume the household's direct utility function is $\mathcal{W}(c_t, 1 - n_t - n_{c,t})$. Substitute for $n_{c,t}$ in $\mathcal{W}(\cdot, 1 - n_t - n_{c,t})$ to construct the indirect utility function

$$\mathcal{W}\left(c_t, 1 - n_t - \mathcal{N}\left(c_t, \frac{M_t}{P_t}\right)\right) = \mathcal{U}\left(c_t, \frac{M_t}{P_t}, n_t\right).$$

The difference between $\mathcal{W}(\cdot, \cdot)$ and $\mathcal{U}(\cdot, \cdot, \cdot)$ is the indirect utility function contains a market price and a nominal stock of wealth, M_t.

Restrictions on the transactions technology translate into restrictions on the MIUF. For example, when $\mathcal{U}_{c,M}$ is positive, additional real balances yield higher marginal utility of consumption. Hence, consumption and real balances are complements. This restriction on the indirect utility function gives MIUF the interpretation that real balances produce transactions services. An implication is an increase in expected inflation lowers real economic activity in response to a higher rate of money growth.

On the other hand, $\mathcal{U}_{c,M} \leq 0$ implies the household is willing to substitute real balances for consumption (or the converse). By making consumption and real balances substitutes, real balances serve as an asset in utility because higher real balances are associated with lower consumption. When money growth generates higher expected inflation, greater real economic activity results under the assumption $\mathcal{U}_{c,M} \leq 0$.

The direct utility function determines the sign of the cross derivative, $\mathcal{U}_{c,M}$. Differentiation with respect to c followed by differentiation with respect to m yields

$$\mathcal{U}_{c,M} \quad = \quad \Big[W_{\ell,\ell}\mathcal{N}_c \; - \; \mathcal{W}_{c,\ell}\Big]\mathcal{N}_M \; - \; \mathcal{W}_{\ell}\mathcal{N}_{c,M}.$$

Theory provides no restrictions to sign $\mathcal{U}_{c,M}$. The lack of information to sign the cross derivatives $\mathcal{W}_{c,\ell}$ and $\mathcal{N}_{c,M}$ suggests MIUF models have free parameters that allow a range of responses to money growth shocks. The problem is there are no a priori restrictions that aid in choosing among these responses. Further, the assumption the transactions technology $\mathcal{C}(\cdot, \cdot)$ depends only on real balances and not other assets begs the question of why only fiat currency is the medium of exchange, especially when innovations are generated by financial markets.

A clear understanding of the relationship between the transaction technology and MIUF models is gained by studying

Exercise Shopping Time: Assume lifetime expected discounted household utility is

$$\mathbf{E}_t\left\{\sum_{j=0}^{\infty}\beta^j W\left(c_{t+j},\; 1 \,-\, n_{t+j} \,-\, \mathcal{N}\left(c_{t+j},\, \frac{M_{t+j}}{P_{t+j}}\right)\right)\right\}.$$

The budget constraint of the household is (10) with the technology (11). All the usual parameter restrictions apply and the stochastic processes of money growth (16) and labor augmenting technical change (17) are given in **Exercise MIUF**.

(i) What restriction (or restrictions) on $\mathcal{N}(\cdot, \cdot)$ necessary to prevent the nominal return on bonds, R_t, from being zero? Interpret the restriction(s).

(ii) Compare and contrast your answer to part (i) with the equilibrium condition (14). [Hint: Focus on the restriction(s) required for money to have positive and finite value in the MIUF model's equilibrium.]

(iii) Provide restrictions on utility to guarantee superneutrality in the steady state of the transactions services model of real balances.

III.A The CIA Constraint

Explaining the existence of fiat currency is a deep and fundamental issue in monetary economics. The question is which frictions (*i.e.*, market incompleteness) in an economy give agents incentives to hold fiat currency. The problem is equivalent to asking why fiat currency has strictly positive and finite value when it is dominated in (nominal) rate of return by other assets. Economists have been studying the primitives responsible for giving fiat currency finite value for a long time. The new monetarist research program seeks to answer these questions. Williamson and Wright (2010a, b) and Logos, Rocheteau, and Wright (2017) survey this research. Also, see Walsh (2017, section 3.4) for an introduction to this class of monetary models.

One restriction that gives fiat currency positive and finite value is the CIA constriant. The CIA constraint appears ad hoc or not a deep primitive. However, Camera and Chien (2016) argue the differences between a monetary model with a CIA constraint and the conical new monetarist model are more similar than not. Under a CIA constraint, the household has to own cash before purchasing goods and services. In this case, barter is not possible.

An inequality that captures this notion is the simple CIA constraint

$$c_t \;\leq\; \frac{M_t}{P_t}. \tag{18}$$

Consumption is constrained to be no more than market value of real balances. The household can use no other real or nominal resources to purchase the consumption good during date t under the CIA constraint (18). There are variations of the CIA constraint that add real wages, and subtract bank deposits or government bonds from the right hand side of the inequality or add investment to the left hand side; see Nason and Cogley (1994) and Belongia and Ireland (2014). However, the central point remains that only the real balances the household owns at the end of date $t-1$ and carry into date t are available to purchase the consumption good.

Lets solve this constrained optimization problem using dynamic programing methods and Bellman's equation. Bellman's equation is

$$\mathcal{J}\left(k_t,\, b_t,\, \frac{M_t}{P_t},\, z_t,\, \mu_t\right) \;=$$

$$\mathbf{Max}_{(c_t, n_t, k_{t+1}, b_{t+1}, M_{t+1})}\left[\mathcal{V}(c_t,\, 1-n_t) \;+\; \beta \mathbf{E}_t\left\{\mathcal{J}\left(k_{t+1},\, b_{t+1},\, \frac{M_{t+1}}{P_{t+1}},\, z_{t+1},\, \mu_{t+1}\right)\right\}\right], \tag{19}$$

subject to the budget constraint (10), the production technology (11), and the CIA constraint (18) given the state variables of the capital stock, the stock of nominal balances taking the real return to the unit discount, the aggregate price level, and the stochastic processes of money growth (16) and labor augmenting technical change (17) parametrically, where $\mathcal{J}\left(k_t,\, b_t,\, \frac{M_t}{P_t},\, z_t,\, \mu_t\right)$ is the value function of the household, and the period utility function of the household, $\mathcal{V}(c_t,\, 1-n_t)$, has standard restrictions. Denote the multipliers of the budget constraint (10) and the CIA constraint (18) by $\lambda_{1,t}$ and $\lambda_{2,t}$, respectively. The latter shadow price represents the transactions services of real balances. The FONCs of the dynamic program (19) are

$$\mathcal{V}_{c,t} \;-\; \lambda_{1,t} \;-\; \lambda_{2,t} \;=\; 0, \tag{20}$$

$$-\mathcal{V}_{n,t} \;+\; \lambda_{1,t}(1 \;-\; \theta)\frac{y_t}{n_t} \;=\; 0, \tag{21}$$

$$-\lambda_{1,t} \;+\; \beta \mathbf{E}_t\left\{\frac{\partial \mathcal{J}_{t+1}}{\partial k_{t+1}}\right\} \;=\; 0, \tag{22}$$

$$-\lambda_{1,t} \;+\; \beta \mathbf{E}_t\left\{\frac{\partial \mathcal{J}_{t+1}}{\partial b_{t+1}}\right\} \;=\; 0, \tag{23}$$

and

$$-\frac{\lambda_{1,t}}{P_t} \;+\; \beta \mathbf{E}_t\left\{\frac{\partial \mathcal{J}_{t+1}}{\partial M_{t+1}}\right\} \;=\; 0. \tag{24}$$

The impact of the CIA constraint (18) is the marginal utility of consumption is the sum of the shadow price of a unit of real income, $\lambda_{1,t}$, plus the shadow price of a unit of real balances, $\lambda_{1,t}$. Rather than a direct utility effect, the CIA approach places a value in the marginal utility of consumption on holding real balances because it is the transactions medium.

Constructing the envelop conditions of the CIA constraint model is an intermediate step between the FONCs and optimality conditions. The envelop conditions are

$$\frac{\partial \mathcal{J}_t}{\partial k_t} = \lambda_{1,t}\left[\theta \frac{y_t}{k_t} + (1 - \delta)\right], \tag{25}$$

$$\frac{\partial \mathcal{J}_t}{\partial b_t} = \lambda_{1,t}(1 + r_t), \tag{26}$$

and

$$\frac{\partial \mathcal{J}_t}{\partial M_t} = \frac{\lambda_{1,t} + \lambda_{2,t}}{P_t}. \tag{27}$$

The current value to the household's program of a unit of real balances is the sum of the shadow prices of the consumption good and real balances valued at the purchasing power of money.

An implication of the CIA constraint is the household treats fiat currency in the same way it does any financial asset. Push the envelope condition (27) ahead one period and combine it with the FONC (24) of money to produce the stochastic difference equation

$$\frac{\lambda_{1,t}}{P_t} = \beta \mathbf{E}_t\left\{\frac{\lambda_{1,t+1} + \lambda_{2,t+1}}{P_{t+1}}\right\}. \tag{28}$$

This is familiar because it generates the solution

$$\frac{\lambda_{1,t}}{P_t} = \mathbf{E}_t\left\{\sum_{j=1}^{\infty} \beta^j \frac{\lambda_{2,t+j}}{P_{t+j}}\right\}. \tag{29}$$

This is the familiar present discounted value asset pricing equation. The transversality condition, $\lim_{j \longrightarrow \infty} \beta^j \mathbf{E}_t\left\{\lambda_{1,t+j}/P_{t+j}\right\} = 0$, is invoked to construct the infinite sum of the present value relation (29).

The present value relation (29) states that the purchasing power of the shadow price of a unit of the consumption today is positive only if the purchasing power of the future stream of the value of the transactions services of real balances is positive. The CIA constraint (18) is expected to bind strictly only when $0 < \lambda_{2,t+j}$, for all $j \geq 0$. The present discounted value relation (29) and the CIA constraint (18) together insure the aggregate price level P_t (or the purchasing power of money) is positive and finite. Nonetheless, if and only if the CIA constraint (18) holds with equality is $P_t \in (0, \infty)$ in equilibrium.

The FONCs connect the left hand side of the present discounted value relation (29) to the marginal utility of consumption, $\mathcal{V}_{c,t}$. Add $\lambda_{2,t}/P_t$ to both sides of the equality of the present discounted value relation (29) and use the FONC (20) of consumption to find

$$\mathcal{V}_{c,t} \;=\; \mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \beta^j \frac{P_t}{P_{t+j}} \lambda_{2,t+j} \right\}.$$

Next, multiple and divide the right hand side of the previous equation by P_{t+j-1} and remember the definition of inflation to show

$$\mathcal{V}_{c,t} \;=\; \mathbf{E}_t \left\{ \sum_{j=0}^{\infty} \Xi_{t+j} \lambda_{2,t+j} \right\}, \quad \Xi_{t+j} \;\equiv\; \beta^j \left(\prod_{i=0}^{j} \frac{1}{1 + \pi_{t+i}} \right), \quad \pi_t \;\equiv\; 0. \tag{30}$$

The present value relation (30) identifies $\mathcal{V}_{c,t}$ with the future path of the value of transactions services of real balances. For the marginal utility of consumption to be positive, the future expected discounted value of the flow of transactions services of real balances must be positive. This flow of transactions services of real balances is discounted at Ξ_{t+j}, which depends on the expected future real return to cash, $1/(1 + \pi_{t+j})$, or the inverse of the inflation rate. The discount rate Ξ_{t+j} is uncertain. Higher future expected inflation rates lower the current marginal utility of consumption because it encourages the household to move consumption forward in time. The expected increase in future inflation acts like a tax on the future value of the transactions services of real balances. The expected inflation tax lowers the stock of future real balances the household anticipates it owns in the future and with it the ability of the household to transact in the future.

This suggests

Exercise CIA Constraint 1: Construct the MIUF version of the present value relation (30). Compare and contrast the two present value relations and their impact on the equilibrium determination of the aggregate price level.

The CIA approach to money imposes an important restriction on the optimality conditions of the economy that distinguishes it from the MIUF approach. The FONCs (20) of consumption and (24) of money and the envelope condition (27) of money yield

$$\lambda_{1,t} \;=\; \beta\,\mathbf{E}_t\left\{\frac{\mathcal{V}_{c,t+1}}{1+\pi_{t+1}}\right\}. \tag{31}$$

This is the forward looking stochastic discount factor (SDF) of the household. Rather than the current marginal utility of consumption equaling $\lambda_{1,t}$, the current shadow price of another unit of real resources is the discounted expected real return to cash between dates t and $t+1$ valued at the date $t+1$ marginal utility of consumption. The CIA constraint (18) limits the household to use an extra unit of cash it obtains today to purchase more of the consumption good tomorrow. Since the purchasing power of money can change between dates t and $t+1$, the SDF depends on the real return to cash (*i.e.*, the inverse of π_{t+1}), which is a random variable.

The SDF affects the optimality conditions of the economy. The FONC (21) of labor becomes

$$\mathcal{V}_{n,t} \;=\; \Gamma_t(1\;-\;\theta)\frac{Y_t}{n_t}, \tag{32}$$

where define the SDF as $\Gamma_t \equiv \lambda_{1,t}$. Unlike the MIUF model, the CIA approach introduces nominal factors into labor market optimality. The marginal rate of substitution between labor and consumption involves explicit intertemporal factors because of the CIA constraint (18. The cash the household garners from current labor income is not available to buy consumption until date $t+1$. The Euler equations for capital and the unit discount bond are found by pushing the envelop conditions (25) and (26) forward one period and substituting the result into the FONCs (22) and (23) to produce

$$\Gamma_t = \beta \mathbf{E}_t \left\{ \Gamma_{t+1} \left[\theta \frac{y_{t+1}}{k_{t+1}} + (1 - \delta) \right] \right\}, \tag{33}$$

and

$$\Gamma_t = \beta \mathbf{E}_t \left\{ \Gamma_{t+1} (1 + r_{t+1}) \right\}. \tag{34}$$

Inspection of the optimality conditions (31), (32), (33), and (34) reveals that superneutrality fails in this economy. The addition of the CIA constraint (18) introduces a welfare cost on the household. It cannot transact in the goods market without cash whenever the CIA constraint binds. Moreover, the household faces an externality, given the CIA constraint (18) binds. The equilibrium aggregate price level and the stock of real balances are determined by other agents, who are other households and the monetary authority.

The monetary propagation mechanism of the CIA model is more highly restricted than it is in the MIUF model. The expected inflation effect drives the monetary business cycle of the CIA model. Assume that utility is separable in consumption and leisure and is a power function in utility. In this case, the Euler equation (34) of bonds is equivalent to

$$\mathbf{E}_t \left\{ \frac{c_{t+1}^{-\alpha}}{1 + \pi_{t+1}} \right\} = \beta \mathbf{E}_t \left\{ \frac{c_{t+2}^{-\alpha}}{1 + \pi_{t+2}} \left(\frac{1 + R_{t+1}}{1 + \pi_{t+1}} \right) \right\}. \tag{35}$$

Remember the Euler equation balances the utility cost of postponing consumption to buy a bond against the utility benefits of future consumption obtained from the income generated by the bond under the CIA constraint (18).

Next, lets review a result about log linearizing Euler equations. Consider the Euler equation

$$1 = \beta \mathbf{E}_t \left\{ \left(\frac{c_{t+1}}{c_t} \right)^{-\alpha} (1 + q_{t+1}) \right\},$$

where q_{t+1} is the return on an asset. Assume consumption growth and asset returns are jointly log normally distributed and homoskedastic. The motivation for the assumption is to link risk aversion and fluctuations in consumption growth to time-variation in expected returns in the intertemporal capital asset pricing model (ICAPM). The ICAPM predicts that consumption growth is the factor that drives time-variation in asset returns. For example, this link (or the lack thereof) is the source of the risk-free rate puzzle (*i.e.*, short rates are low and lack volatility relative to the predictions of the ICAPM). Log linearizing the Euler equation and applying the property of log normality and homoskedasticity results in

$$0 = \ln\beta - \alpha\mathbf{E}_t\{g_{c,t+1}\} + \mathbf{E}_t\{q_{t+1}\} + \frac{1}{2}\left[\alpha^2\sigma^2_{g_c} + \sigma^2_q\right] - \alpha\,\mathrm{Cov}(g_{c,t+1}, q_{t+1}),$$

where $\ln(1+q_t) \approx q_t$, $g_{c,t} = \ln(c_t/c_{t-1})$, and $\ln\mathbf{E}_t\{x_{t+1}\} = \mathbf{E}_t\{\ln x_{t+1}\} + 0.5\mathrm{Var}(\ln x_{t+1})$ because of log normality and homoskedasticity. Since the risk-free rate is the zero-risk asset and uncorrelated with $g_{c,t+1}$, the linearized Euler equation for the risk-free rate is

$$\mathbf{E}_t\{r_{F,t+1}\} = -\ln\beta + \alpha\mathbf{E}_t\{g_{c,t+1}\} - \frac{\alpha^2}{2}\sigma^2_{g_c}. \tag{36}$$

This is the expected risk-free return generating equation. It predicts consumption growth is the lone factor producing interest rate fluctuations. A one percent change in consumption growth is expected to produce an α percent change in $r_{F,t+1}$.

These results lead to the log linearized version of the Euler equation (35)

$$\mathbf{E}_t\{R_{t+1}\} = \alpha\mathbf{E}_t\{g_{c,t+1}\} + \mathbf{E}_t\{\pi_{t+2}\},$$

where constants are ignored. If the expected risk-free rate is uncorrelated with expected consumption growth (see the generating equation (36) of $\mathbf{E}_t\{r_{F,t+1}\}$), the Fisher's equation of the CIA model depends on expectations of inflation at date $t+2$. Substitute for $\mathbf{E}_t\{g_{c,t+1}\}$ using equation (36) and add and subtract $\mathbf{E}_t\{\pi_{t+1}\}$ in the log linearized Euler equation to show

$$\mathbf{E}_t\{R_{t+1}\} = \mathbf{E}_t\{r_{F,t+1}\} + \mathbf{E}_t\{\pi_{t+1}\} + \mathbf{E}_t\{g_{\pi,t+2}\}, \tag{37}$$

where $g_{\pi,t+2} = \pi_{t+2} - \pi_{t+1}$ and constants are ignored. In the CIA model, the expected inflation effect is reinforced by potential persistence in inflation growth. Unlike the Fisher equation (15) of the MIUF model, there is no chance the liquidity effect operates in the CIA model.

This is the good moment to ask

Exercise CIA Constraint 2: Let the period utility of the household be

$$\mathcal{V}(c_t,\, 1 - n_t) \;=\; \frac{c_t^{1-\alpha}}{1-\alpha}\left(\frac{\ell_t^{1-\nu}}{1-\nu}\right),$$

with all the usual restrictions. The budget constraint of the household is (10), technology given by the CRS production function (11), and the CIA constraint is equation (18). Once again, all the usual parameter restrictions apply and the stochastic processes of money growth (16) and labor augmenting TFP (17) are given in exercise MIUF. To detrend Γ_t, work inside the expectation operator of the stochastic discount factor (31) to see that $\widetilde{\Gamma}_t = \Gamma_t\, z_t^{-\alpha}$.

(i) Construct the optimality conditions of the economy. Interpret these equations.

(ii) Show superneutrality fails to hold in the steady state equilibrium of this model. Without re-solving the model, discuss the impact of utility non-separable in consumption and leisure on the steady state and superneutrality in this CIA model. In the MIUF economy, log utility permits superneutrality result. Does this result hold in this CIA model? Explain.

(iii) Show the inflation tax lowers the real return to capital in this economy. Discuss the restrictions on utility, technology, and the technology and money growth shocks that raise the cost of the inflation tax in this CIA model.

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